



1 Write each quantity in SI units.

- a. A speed of 90 km h^{-1}
- b. A mass of 1.2 tonnes
- c. A speed of 45 cm per minute

(3 marks)

a) km $\rightarrow$ m and hours $\rightarrow$ seconds

$$90 \text{ km h}^{-1} = \frac{90\,000}{3600} \text{ m s}^{-1}$$

$$= 25 \text{ m s}^{-1}$$

25 m s^{-1}

b) 1 tonne = 1000 kg

$$1.2 \text{ tonnes} = 1.2 \times 1000 \text{ kg}$$

$$= 1200 \text{ kg}$$

1200 kg

c) cm $\rightarrow$ m and minutes $\rightarrow$ seconds

$$45 \text{ cm} = 0.45 \text{ m}$$

$$1 \text{ minute} = 60 \text{ s}$$

$$\text{speed} = \frac{0.45}{60}$$

$$= 0.0075 \text{ m s}^{-1}$$

0.0075 m s^{-1}

2

A parcel is pulled across a floor by a rope. In a model of the motion the parcel is modelled as a particle, the rope is modelled as light and inextensible, and the floor is modelled as smooth.

State what each of these words means for the model.

- a. particle
- b. light
- c. inextensible
- d. smooth

(4 marks)

a) Particle

Size of the parcel is ignored

all its mass acts at one point, no rotation

b) Light

Mass of the rope is ignored

so tension is the same all along the rope

c) Inextensible

The rope does not stretch

so the parcel has the same acceleration as the rope

d) Smooth

No friction between the parcel and the floor

3

Amara walks 300 m due east and then 400 m due north. The whole walk takes 500 s.

- Find the total distance Amara walks.
- Find the magnitude of her displacement from her starting point, and its bearing.
- Find her average speed and the magnitude of her average velocity.

(5 marks)

a) Distance = length of the path

$$\begin{aligned}\text{distance} &= 300 + 400 \\ &= 700 \text{ m}\end{aligned}$$

700 m

b) Displacement = straight line from start to end

$$\begin{aligned}|\text{displacement}| &= \sqrt{(300^2 + 400^2)} \\ &= \sqrt{250\,000} \\ &= 500 \text{ m}\end{aligned}$$

Angle from north: 300 east, 400 north

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{300}{400}\right) \\ &= 36.9^\circ\end{aligned}$$

500 m on a bearing of 037°

c) Average speed uses distance

$$\begin{aligned}\text{average speed} &= \frac{700}{500} \\ &= 1.4 \text{ m s}^{-1}\end{aligned}$$

Average velocity uses displacement

$$\begin{aligned}|\text{average velocity}| &= \frac{500}{500} \\ &= 1 \text{ m s}^{-1}\end{aligned}$$

1.4 m s⁻¹ and 1 m s⁻¹

4

In this question $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north.

A boat moves with constant velocity $(5\mathbf{i} - 12\mathbf{j}) \text{ m s}^{-1}$.

a. Find the speed of the boat.

b. Find the direction in which the boat is moving, as a bearing to the nearest degree.

(4 marks)

a) Speed = magnitude of velocity

$$\begin{aligned}\text{speed} &= \sqrt{5^2 + (-12)^2} \\ &= \sqrt{169} \\ &= 13 \text{ m s}^{-1}\end{aligned}$$

$$\boxed{13 \text{ m s}^{-1}}$$

b) Sketch: 5 east, 12 south

angle below east is θ

$$\begin{aligned}\tan \theta &= \frac{12}{5} \\ \theta &= 67.38^\circ\end{aligned}$$

Bearing is measured clockwise from north

$$\begin{aligned}\text{bearing} &= 90^\circ + 67.38^\circ \\ &= 157.38^\circ\end{aligned}$$

$$\boxed{157^\circ}$$

5

A particle P moves with constant velocity $(4\mathbf{i} - \mathbf{j}) \text{ m s}^{-1}$. At time $t = 0$, P is at the point with position vector $(2\mathbf{i} + 3\mathbf{j}) \text{ m}$ relative to a fixed origin O .

- Find the position vector of P when $t = 5$
- Find the value of t when P is due east of O .
- Find the distance OP at this time.

(5 marks)

Constant velocity: $r = r_0 + vt$

$$r = (2\mathbf{i} + 3\mathbf{j}) + t(4\mathbf{i} - \mathbf{j})$$

$$r = (2 + 4t)\mathbf{i} + (3 - t)\mathbf{j}$$

a) $t = 5$

$$\begin{aligned} r &= (2 + 20)\mathbf{i} + (3 - 5)\mathbf{j} \\ &= 22\mathbf{i} - 2\mathbf{j} \end{aligned}$$

$$\boxed{(22\mathbf{i} - 2\mathbf{j}) \text{ m}}$$

b) Due east of O : $\mathbf{j}$ component = 0

$$3 - t = 0$$

$$t = 3$$

$$\boxed{t = 3}$$

c) Substitute $t = 3$

$$\begin{aligned} r &= (2 + 12)\mathbf{i} + 0\mathbf{j} \\ &= 14\mathbf{i} \end{aligned}$$

$$OP = 14 \text{ m}$$

$$\boxed{14 \text{ m}}$$

6

In this question $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north, and position vectors are given relative to a fixed harbour H .

A ship moves with constant velocity. At 09:00 it is at the point with position vector $(3\mathbf{i} + 7\mathbf{j})$ km. At 11:00 it is at the point with position vector $(15\mathbf{i} - 2\mathbf{j})$ km.

- Find the velocity of the ship in km h^{-1} .
- Find the speed of the ship.
- Find the bearing on which the ship is moving.

(6 marks)

a) Displacement from 09:00 to 11:00

$$\begin{aligned} \mathbf{d} &= (15\mathbf{i} - 2\mathbf{j}) - (3\mathbf{i} + 7\mathbf{j}) \\ &= 12\mathbf{i} - 9\mathbf{j} \end{aligned}$$

Time taken = 2 hours

$$\begin{aligned} \mathbf{v} &= \frac{12\mathbf{i} - 9\mathbf{j}}{2} \\ &= 6\mathbf{i} - 4.5\mathbf{j} \end{aligned}$$

$$(6\mathbf{i} - 4.5\mathbf{j}) \text{ km h}^{-1}$$

b) Speed = magnitude

$$\begin{aligned} \text{speed} &= \sqrt{6^2 + 4.5^2} \\ &= \sqrt{56.25} \\ &= 7.5 \text{ km h}^{-1} \end{aligned}$$

$$7.5 \text{ km h}^{-1}$$

c) 6 east, 4.5 south: angle below east

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{4.5}{6}\right) \\ &= 36.87^\circ \end{aligned}$$

$$\begin{aligned} \text{bearing} &= 90^\circ + 36.87^\circ \\ &= 126.87^\circ \end{aligned}$$

$$127^\circ$$

7

A particle moves with velocity $(3k\mathbf{i} + (k - 5)\mathbf{j}) \text{ m s}^{-1}$, where k is a constant.

Given that the particle is moving parallel to the vector $(\mathbf{i} - \mathbf{j})$,

- find the value of k ,
- find the speed of the particle, giving your answer to 3 significant figures.

(5 marks)

a) Parallel to $\mathbf{i} - \mathbf{j}$: velocity $= \lambda(\mathbf{i} - \mathbf{j})$

so $\mathbf{j}$ component $= -(\mathbf{i}$ component)

$$k - 5 = -3k$$

$$4k = 5$$

$$k = 1.25$$

$$\boxed{k = 1.25}$$

b) Substitute $k = 1.25$

$$\mathbf{v} = 3.75\mathbf{i} + (1.25 - 5)\mathbf{j}$$

$$= 3.75\mathbf{i} - 3.75\mathbf{j}$$

$$\text{speed} = \sqrt{(3.75)^2 + (-3.75)^2}$$

$$= 3.75\sqrt{2}$$

$$= 5.303\dots$$

$$\boxed{5.30 \text{ m s}^{-1}}$$

8

Two forces $\mathbf{F}_1 = (3\mathbf{i} - 2\mathbf{j})$ N and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j})$ N are the only forces acting on a particle of mass 2 kg.

The acceleration of the particle is $(4\mathbf{i} + 5\mathbf{j})$ m s⁻².

- Find the values of p and q .
- Find the magnitude of the acceleration.
- Find the angle between the acceleration and the vector $\mathbf{i}$.

(6 marks)

a) Newton's second law: $\mathbf{F} = m\mathbf{a}$

$$\mathbf{F}_1 + \mathbf{F}_2 = 2(4\mathbf{i} + 5\mathbf{j})$$

$$(3 + p)\mathbf{i} + (-2 + q)\mathbf{j} = 8\mathbf{i} + 10\mathbf{j}$$

Compare i components

$$3 + p = 8$$

$$p = 5$$

Compare j components

$$-2 + q = 10$$

$$q = 12$$

$$p = 5, q = 12$$

b) Magnitude of acceleration

$$|\mathbf{a}| = \sqrt{4^2 + 5^2}$$

$$= \sqrt{41}$$

$$= 6.403...$$

$$6.40 \text{ m s}^{-2}$$

c) Angle with i

$$\theta = \tan^{-1}\left(\frac{5}{4}\right)$$

$$= 51.34^\circ$$

$$51.3^\circ$$

9

A sledge is pulled across a frozen lake by a rope held by a child. In a simple model the sledge is a particle, the ice is smooth and the rope is light and inextensible. The model is used to predict the acceleration of the sledge.

- Explain why the acceleration predicted by the model is likely to be greater than the actual acceleration.
- Give one other limitation of modelling the sledge as a particle.
- Suggest one refinement to the model that would make the prediction more accurate.

(4 marks)

a) What does "smooth" leave out?

Real ice has some friction

and the model ignores air resistance

so the resultant force in the model is too big

$F = ma$, so the predicted acceleration is too big

b) Particle

Size and shape of the sledge are ignored

so air resistance and any rotation are not modelled

c) Refinement

Include a friction force between the sledge and the ice
(or include air resistance)

10

In this question $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north, and position vectors are relative to a fixed point O .

At noon boat A is at O and boat B is at the point with position vector $(10\mathbf{i} + 2\mathbf{j})$ km. Boat A moves with constant velocity $(3\mathbf{i} + 4\mathbf{j})$ km h^{-1} and boat B moves with constant velocity $(-2\mathbf{i} + 5\mathbf{j})$ km h^{-1} .

- Find the distance between the boats at noon.
- Show that, t hours after noon, the position vector of B relative to A is $((10 - 5t)\mathbf{i} + (2 + t)\mathbf{j})$ km.
- Find the time at which B is due north of A , and the distance between the boats at that time.

(7 marks)

a) Distance at noon

$$\begin{aligned} |10\mathbf{i} + 2\mathbf{j}| &= \sqrt{10^2 + 2^2} \\ &= \sqrt{104} \\ &= 10.198... \end{aligned}$$

10.2 km

b) Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of A and B

Use $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ for each boat

$$\mathbf{a} = 3t\mathbf{i} + 4t\mathbf{j}$$

$$\mathbf{b} = (10 - 2t)\mathbf{i} + (2 + 5t)\mathbf{j}$$

B relative to A

$$\begin{aligned} \mathbf{b} - \mathbf{a} &= (10 - 2t - 3t)\mathbf{i} + (2 + 5t - 4t)\mathbf{j} \\ &= (10 - 5t)\mathbf{i} + (2 + t)\mathbf{j} \end{aligned}$$

as required

c) B due north of A : $\mathbf{i}$ component = 0

$$10 - 5t = 0$$

$$t = 2$$

2 hours after noon: 14:00

Substitute $t = 2$

$$\mathbf{b} - \mathbf{a} = 0\mathbf{i} + 4\mathbf{j}$$

distance = 4 km

14:00, 4 km apart

11

A drone flies in a straight line with constant velocity. At time $t = 0$ it is at the point with position vector $(-4\mathbf{i} + 6\mathbf{j})$ m relative to its controller at O . At $t = 5$ s it is at the point with position vector $(16\mathbf{i} - 9\mathbf{j})$ m.

- Find the velocity of the drone and its speed.
- Show that the square of the distance of the drone from O at time t seconds is $25t^2 - 68t + 52$
- By completing the square, find the shortest distance between the drone and O , and the time at which it happens.

(8 marks)

a) Displacement in 5 s

$$\begin{aligned} \mathbf{d} &= (16\mathbf{i} - 9\mathbf{j}) - (-4\mathbf{i} + 6\mathbf{j}) \\ &= 20\mathbf{i} - 15\mathbf{j} \\ \mathbf{v} &= \frac{20\mathbf{i} - 15\mathbf{j}}{5} \\ &= 4\mathbf{i} - 3\mathbf{j} \\ \text{speed} &= \sqrt{4^2 + 3^2} \\ &= 5 \text{ m s}^{-1} \end{aligned}$$

$$(4\mathbf{i} - 3\mathbf{j}) \text{ m s}^{-1}, 5 \text{ m s}^{-1}$$

b) Position at time t

$$\begin{aligned} \mathbf{r} &= (-4\mathbf{i} + 6\mathbf{j}) + t(4\mathbf{i} - 3\mathbf{j}) \\ &= (4t - 4)\mathbf{i} + (6 - 3t)\mathbf{j} \end{aligned}$$

Square of the distance

$$\begin{aligned} |\mathbf{r}|^2 &= (4t - 4)^2 + (6 - 3t)^2 \\ &= 16t^2 - 32t + 16 + 36 - 36t + 9t^2 \\ &= 25t^2 - 68t + 52 \end{aligned}$$

as required

c) Complete the square

$$\begin{aligned} 25t^2 - 68t + 52 &= 25(t^2 - 2.72t) + 52 \\ &= 25[(t - 1.36)^2 - 1.8496] + 52 \\ &= 25(t - 1.36)^2 - 46.24 + 52 \\ &= 25(t - 1.36)^2 + 5.76 \end{aligned}$$

Least value when the bracket is 0

$$\begin{aligned} t &= 1.36 \text{ s} \\ \text{Least } |\mathbf{r}|^2 &= 5.76 \end{aligned}$$

$$\begin{aligned}\text{least distance} &= \sqrt{5.76} \\ &= 2.4 \text{ m}\end{aligned}$$

$$2.4 \text{ m when } t = 1.36 \text{ s}$$

12

In this question $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north.

On a training run, Kwame runs 120 m on a bearing of 060° in 30 s, and then 80 m due south in 20 s.

- Find his displacement from his starting point as a vector in terms of $\mathbf{i}$ and $\mathbf{j}$, with numbers to 3 significant figures.
- Find the distance and bearing of his finishing point from his starting point.
- Find his average speed and the magnitude of his average velocity for the whole run.
- Explain why these two answers are different.

(8 marks)

a) Resolve the first leg (60° from north)

$$\text{east} = 120 \sin 60^\circ$$

$$= 103.92$$

$$\text{north} = 120 \cos 60^\circ$$

$$= 60$$

Second leg is 80 m south

$$\mathbf{d} = 103.92\mathbf{i} + (60 - 80)\mathbf{j}$$

$$= 103.92\mathbf{i} - 20\mathbf{j}$$

$$\boxed{(104\mathbf{i} - 20\mathbf{j}) \text{ m}}$$

b) Magnitude

$$|\mathbf{d}| = \sqrt{103.92^2 + 20^2}$$

$$= \sqrt{11200}$$

$$= 105.8 \text{ m}$$

Angle below east

$$\theta = \tan^{-1}\left(\frac{20}{103.92}\right)$$

$$= 10.89^\circ$$

$$\text{bearing} = 90^\circ + 10.89^\circ$$

$$= 100.9^\circ$$

$$\boxed{106 \text{ m on a bearing of } 101^\circ}$$

c) Total time = $30 + 20 = 50 \text{ s}$

$$\text{average speed} = \frac{120 + 80}{50}$$

$$= 4 \text{ m s}^{-1}$$

$$|\text{average velocity}| = \frac{105.8}{50}$$

$$= 2.12 \text{ m s}^{-1}$$

$$4 \text{ m s}^{-1} \text{ and } 2.12 \text{ m s}^{-1}$$

d) Compare

Speed uses total distance: 200 m

Velocity uses displacement: 106 m

He changed direction, so these are not equal