



- 1** [In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

A particle P moves with constant acceleration $(2\mathbf{i} - 3\mathbf{j}) \text{ m s}^{-2}$. At time $t = 0$ its velocity is $(4\mathbf{i} + \mathbf{j}) \text{ m s}^{-1}$.

- Find the velocity of P when $t = 3$
- Find the speed of P when $t = 3$, giving your answer as an exact surd.

(4 marks)

a) Use $\mathbf{v} = \mathbf{u} + \mathbf{at}$ with vectors

$$\mathbf{v} = (4\mathbf{i} + \mathbf{j}) + 3(2\mathbf{i} - 3\mathbf{j})$$

$$\mathbf{v} = (4 + 6)\mathbf{i} + (1 - 9)\mathbf{j}$$

$$\mathbf{v} = (10\mathbf{i} - 8\mathbf{j}) \text{ m s}^{-1}$$

b) Speed is the magnitude of $\mathbf{v}$

$$|\mathbf{v}| = \sqrt{10^2 + (-8)^2}$$

$$|\mathbf{v}| = \sqrt{164}$$

$$2\sqrt{41} \text{ m s}^{-1} (= 12.8)$$

2

[In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

A particle moves with constant acceleration $(0.5\mathbf{i} + \mathbf{j}) \text{ m s}^{-2}$. At time $t = 0$ it is at the point with position vector $(2\mathbf{i} + 5\mathbf{j}) \text{ m}$ and has velocity $(3\mathbf{i} - 2\mathbf{j}) \text{ m s}^{-1}$.

- Find the position vector of the particle when $t = 4$
- Find the distance of the particle from O when $t = 4$

(4 marks)

a) Displacement: $s = ut + \frac{1}{2}at^2$

$$s = 4(3\mathbf{i} - 2\mathbf{j}) + \frac{1}{2}(16)(0.5\mathbf{i} + \mathbf{j})$$

$$s = (12\mathbf{i} - 8\mathbf{j}) + (4\mathbf{i} + 8\mathbf{j})$$

$$s = 16\mathbf{i}$$

Add the starting position

$$r = (2\mathbf{i} + 5\mathbf{j}) + 16\mathbf{i}$$

$$r = (18\mathbf{i} + 5\mathbf{j}) \text{ m}$$

b) Distance from O is $|r|$

$$|r| = \sqrt{18^2 + 5^2}$$

$$|r| = \sqrt{349} = 18.681\dots$$

$$18.7 \text{ m}$$

3

[In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

At time t seconds, the position vector of a particle P is $\mathbf{r}$ metres, where

$$\mathbf{r} = (t^3 - 4t)\mathbf{i} + (2t^2 + 1)\mathbf{j}$$

a. Find the speed of P when $t = 2$

b. Find the acceleration of P when $t = 2$

(5 marks)

a) Differentiate $\mathbf{r}$ to get $\mathbf{v}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (3t^2 - 4)\mathbf{i} + 4t\mathbf{j}$$

$$t = 2$$

$$\mathbf{v} = (12 - 4)\mathbf{i} + 8\mathbf{j} = 8\mathbf{i} + 8\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{8^2 + 8^2} = \sqrt{128}$$

$$\boxed{8\sqrt{2} \text{ m s}^{-1}}$$

b) Differentiate $\mathbf{v}$ to get $\mathbf{a}$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 6t\mathbf{i} + 4\mathbf{j}$$

$$t = 2$$

$$\boxed{\mathbf{a} = (12\mathbf{i} + 4\mathbf{j}) \text{ m s}^{-2}}$$

4

[In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively, and position vectors are given relative to a fixed origin O .]

A particle P moves with constant acceleration. At time $t = 0$ the velocity of P is $(-3\mathbf{i} + 6\mathbf{j}) \text{ m s}^{-1}$ and at time $t = 4$ seconds its velocity is $(5\mathbf{i} - 2\mathbf{j}) \text{ m s}^{-1}$.

- Find the acceleration of P .
- Find the value of t when P is moving parallel to $\mathbf{i}$.
- Find the value of t when P is moving north-east.

(7 marks)

$$a) v = u + at, \text{ so } a = \frac{v - u}{t}$$

$$a = \frac{(5\mathbf{i} - 2\mathbf{j}) - (-3\mathbf{i} + 6\mathbf{j})}{4}$$

$$a = \frac{8\mathbf{i} - 8\mathbf{j}}{4}$$

$$a = (2\mathbf{i} - 2\mathbf{j}) \text{ m s}^{-2}$$

b) Velocity at time t

$$v = (-3\mathbf{i} + 6\mathbf{j}) + t(2\mathbf{i} - 2\mathbf{j})$$

$$v = (-3 + 2t)\mathbf{i} + (6 - 2t)\mathbf{j}$$

Parallel to $\mathbf{i}$: $\mathbf{j}$ component is zero

$$6 - 2t = 0$$

$$t = 3$$

c) North-east: $\mathbf{i}$ and $\mathbf{j}$ components equal (and positive)

$$-3 + 2t = 6 - 2t$$

$$4t = 9$$

$$t = \frac{9}{4}$$

Check: $v = 1.5\mathbf{i} + 1.5\mathbf{j}$, both positive

$$t = \frac{9}{4}$$

5

[In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively, and position vectors are given relative to a fixed origin O .]

A particle P moves with constant acceleration $(-2\mathbf{i} + \mathbf{j}) \text{ m s}^{-2}$. At time $t = 0$, P is at the point with position vector $(-10\mathbf{i} + 4\mathbf{j}) \text{ m}$ and is moving with velocity $(7\mathbf{i} - 2\mathbf{j}) \text{ m s}^{-1}$.

- Find the position vector of P at time t seconds.
- Find the two values of t for which P is due north of O .
- Show that at the first of these times P is moving due east.

(7 marks)

$$a) \mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = (-10\mathbf{i} + 4\mathbf{j}) + t(7\mathbf{i} - 2\mathbf{j}) + \frac{1}{2}t^2(-2\mathbf{i} + \mathbf{j})$$

$$\mathbf{r} = (-10 + 7t - t^2)\mathbf{i} + (4 - 2t + \frac{1}{2}t^2)\mathbf{j}$$

b) Due north of O : i component is zero (and j component positive)

$$-10 + 7t - t^2 = 0$$

$$t^2 - 7t + 10 = 0$$

$$(t - 2)(t - 5) = 0$$

Check the j components

$$t = 2: 4 - 4 + 2 = 2 > 0$$

$$t = 5: 4 - 10 + 12.5 = 6.5 > 0$$

$$t = 2 \text{ and } t = 5$$

c) $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v} = (7 - 2t)\mathbf{i} + (-2 + t)\mathbf{j}$$

$$t = 2: \mathbf{v} = 3\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{v} = 3\mathbf{i}, \text{ so } P \text{ is moving due east}$$

6

[In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

At time t seconds, where $t \geq 0$, a particle P has velocity $\mathbf{v} \text{ m s}^{-1}$, where

$$\mathbf{v} = (t^2 - 4t + 3)\mathbf{i} + (2t + 1)\mathbf{j}$$

- Find the values of t when P is moving parallel to $\mathbf{j}$.
- Find the magnitude of the acceleration of P when $t = 4$
- Find the displacement of P from $t = 0$ to $t = 3$

(8 marks)

a) Parallel to $\mathbf{j}$: $\mathbf{i}$ component is zero

$$t^2 - 4t + 3 = 0$$

$$(t - 1)(t - 3) = 0$$

$\mathbf{j}$ component $2t + 1$ is not zero at these times

$$t = 1 \text{ and } t = 3$$

b) Differentiate $\mathbf{v}$

$$\mathbf{a} = (2t - 4)\mathbf{i} + 2\mathbf{j}$$

$$t = 4: \mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$$

$$|\mathbf{a}| = \sqrt{(4^2 + 2^2)} = \sqrt{20}$$

$$2\sqrt{5} \text{ m s}^{-2}$$

c) Integrate $\mathbf{v}$

$$\int \mathbf{v} \, dt = \left(\frac{1}{3}t^3 - 2t^2 + 3t\right)\mathbf{i} + (t^2 + t)\mathbf{j}$$

Limits: [...] from 0 to 3

$$\mathbf{i}: 9 - 18 + 9 - 0 = 0$$

$$\mathbf{j}: 9 + 3 - 0 = 12$$

$$\text{displacement} = 12\mathbf{j} \text{ m}$$

7

[In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

At time t seconds, where $t \geq 0$, a particle P has acceleration $\mathbf{a} \text{ m s}^{-2}$, where

$$\mathbf{a} = 3t^{\frac{1}{2}}\mathbf{i} - 2\mathbf{j}$$

When $t = 0$, P is at O and is moving with velocity $(4\mathbf{i} + 5\mathbf{j}) \text{ m s}^{-1}$.

- Find the speed of P when $t = 4$
- Find the position vector of P when $t = 4$

(8 marks)

a) Integrate $\mathbf{a}$

$$\mathbf{v} = 3 \times \frac{2}{3} t^{3/2} \mathbf{i} - 2t \mathbf{j} + \mathbf{c}$$

$$\mathbf{v} = 2t^{3/2} \mathbf{i} - 2t \mathbf{j} + \mathbf{c}$$

$t = 0$: $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j}$, so $\mathbf{c} = 4\mathbf{i} + 5\mathbf{j}$

$$\mathbf{v} = (2t^{3/2} + 4)\mathbf{i} + (5 - 2t)\mathbf{j}$$

$t = 4$, and $4^{3/2} = 8$

$$\mathbf{v} = 20\mathbf{i} - 3\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{(20^2 + 3^2)} = \sqrt{409}$$

$$|\mathbf{v}| = 20.223...$$

$$\boxed{20.2 \text{ m s}^{-1}}$$

b) Integrate $\mathbf{v}$

$$\mathbf{r} = \left(\frac{2}{5/2} t^{5/2} + 4t \right) \mathbf{i} + (5t - t^2) \mathbf{j} + \mathbf{d}$$

$$\mathbf{r} = \left(\frac{4}{5} t^{5/2} + 4t \right) \mathbf{i} + (5t - t^2) \mathbf{j} + \mathbf{d}$$

$t = 0$: $\mathbf{r} = \mathbf{0}$, so $\mathbf{d} = \mathbf{0}$

$t = 4$, and $4^{5/2} = 32$

$$\mathbf{r} = \left(\frac{4}{5}(32) + 16 \right) \mathbf{i} + (20 - 16) \mathbf{j}$$

$$\boxed{\mathbf{r} = (41.6\mathbf{i} + 4\mathbf{j}) \text{ m}}$$

8

[In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

Two particles A and B move on a smooth horizontal surface with constant velocities. At time $t = 0$, A is at the point with position vector $(2\mathbf{i} + \mathbf{j})$ m and B is at the point with position vector $(10\mathbf{i} - 3\mathbf{j})$ m. The velocity of A is $(3\mathbf{i} + \mathbf{j})$ m s⁻¹ and the velocity of B is $(-\mathbf{i} + 3\mathbf{j})$ m s⁻¹.

- Find the vector $\overline{AB}$ in terms of t .
- Show that A and B collide, and find the position vector of the point where they collide.
- Find the exact distance between A and B when $t = 0.5$

(7 marks)

a) Position vectors at time t

$$r_A = (2 + 3t)\mathbf{i} + (1 + t)\mathbf{j}$$

$$r_B = (10 - t)\mathbf{i} + (-3 + 3t)\mathbf{j}$$

$$\overline{AB} = r_B - r_A$$

$$\overline{AB} = (8 - 4t)\mathbf{i} + (2t - 4)\mathbf{j}$$

b) Collide if $\overline{AB} = 0$ at the same time

$$8 - 4t = 0 \text{ gives } t = 2$$

$$2t - 4 = 0 \text{ gives } t = 2$$

Same t , so they collide

$$r_A = (2 + 6)\mathbf{i} + (1 + 2)\mathbf{j}$$

$$\text{they collide at } (8\mathbf{i} + 3\mathbf{j}) \text{ m}$$

c) Substitute $t = 0.5$ into $\overline{AB}$

$$\overline{AB} = (8 - 2)\mathbf{i} + (1 - 4)\mathbf{j}$$

$$\overline{AB} = 6\mathbf{i} - 3\mathbf{j}$$

$$|\overline{AB}| = \sqrt{36 + 9} = \sqrt{45}$$

$$3\sqrt{5} \text{ m}$$

[In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular horizontal unit vectors and position vectors are given relative to a fixed origin O .]

A particle P moves with constant acceleration $(2\mathbf{i} - \mathbf{j}) \text{ m s}^{-2}$. At time $t = 0$, P passes through O with velocity $(-4\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-1}$. At time T seconds, where $T > 0$, P passes through the point C with position vector $(\lambda\mathbf{i} - 6\mathbf{j}) \text{ m}$, where λ is a constant.

- Find the value of T .
- Find the value of λ .
- Find the exact speed of P as it passes through C .

(8 marks)

$$\mathbf{r} = \mathbf{ut} + \frac{1}{2}\mathbf{at}^2 \text{ (starts at } O \text{)}$$

$$\mathbf{r} = t(-4\mathbf{i} + 2\mathbf{j}) + \frac{1}{2}t^2(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{r} = (-4t + t^2)\mathbf{i} + (2t - \frac{1}{2}t^2)\mathbf{j}$$

a) Match the $\mathbf{j}$ component with -6

$$2T - \frac{1}{2}T^2 = -6$$

$$T^2 - 4T - 12 = 0$$

$$(T - 6)(T + 2) = 0$$

$$T > 0$$

$$\boxed{T = 6}$$

b) $\mathbf{i}$ component at $T = 6$

$$\lambda = -4(6) + 6^2$$

$$\lambda = -24 + 36$$

$$\boxed{\lambda = 12}$$

c) $\mathbf{v} = \mathbf{u} + \mathbf{at}$ at $t = 6$

$$\mathbf{v} = (-4 + 12)\mathbf{i} + (2 - 6)\mathbf{j}$$

$$\mathbf{v} = 8\mathbf{i} - 4\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{64 + 16} = \sqrt{80}$$

$$\boxed{4\sqrt{5} \text{ m s}^{-1}}$$

10

[In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively, and position vectors are given relative to a fixed origin O .]

At time t seconds, where $t \geq 0$, the position vector of a particle P is $\mathbf{r}$ metres, where

$$\mathbf{r} = (t^3 - 3t)\mathbf{i} + (5t - t^2)\mathbf{j}$$

- Find the velocity of P at time t .
- Find the value of t when P is moving north-east.
- Find the distance of P from O when P is due east of O .
- Find the magnitude of the acceleration of P at this time.

(9 marks)

a) Differentiate $\mathbf{r}$

$$\mathbf{v} = (3t^2 - 3)\mathbf{i} + (5 - 2t)\mathbf{j}$$

b) North-east: $\mathbf{i}$ and $\mathbf{j}$ components equal and positive

$$3t^2 - 3 = 5 - 2t$$

$$3t^2 + 2t - 8 = 0$$

$$(3t - 4)(t + 2) = 0$$

$$t = \frac{4}{3} \quad (t \geq 0)$$

Check: $\mathbf{v} = \frac{7}{3}\mathbf{i} + \frac{7}{3}\mathbf{j}$, both positive

$$t = \frac{4}{3}$$

c) Due east of O : $\mathbf{j}$ component zero, $\mathbf{i}$ component positive

$$5t - t^2 = 0$$

$$t(5 - t) = 0$$

$$t = 5 \quad (t = 0 \text{ gives } P \text{ at } O)$$

$$\mathbf{i} \text{ component: } 125 - 15 = 110$$

$$110 \text{ m}$$

d) Differentiate $\mathbf{v}$

$$\mathbf{a} = 6t\mathbf{i} - 2\mathbf{j}$$

$$t = 5: \mathbf{a} = 30\mathbf{i} - 2\mathbf{j}$$

$$|\mathbf{a}| = \sqrt{(900 + 4)} = \sqrt{904}$$

$$|\mathbf{a}| = 30.066...$$

$$30.1 \text{ m s}^{-2}$$

[In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively, and position vectors are given relative to a fixed origin O .]

At noon, ship A is at O and ship B is at the point with position vector $(20\mathbf{i} + 4\mathbf{j})$ km. Ship A moves with constant velocity $(6\mathbf{i} + 8\mathbf{j})$ km h^{-1} and ship B moves with constant velocity $(2\mathbf{i} + 10\mathbf{j})$ km h^{-1} . At time t hours after noon the position vectors of A and B are $\mathbf{r}_A$ km and $\mathbf{r}_B$ km.

- Show that the position vector of B relative to A is $(20 - 4t)\mathbf{i} + (4 + 2t)\mathbf{j}$
- Find the time at which B is due north of A , and the distance between the ships at that time.
- Find, using calculus, the time at which the ships are closest together and the least distance between them.

(9 marks)

a) Position vectors at time t

$$\mathbf{r}_A = 6t\mathbf{i} + 8t\mathbf{j}$$

$$\mathbf{r}_B = (20 + 2t)\mathbf{i} + (4 + 10t)\mathbf{j}$$

$$\mathbf{r}_B - \mathbf{r}_A = (20 + 2t - 6t)\mathbf{i} + (4 + 10t - 8t)\mathbf{j}$$

$$\mathbf{r}_B - \mathbf{r}_A = (20 - 4t)\mathbf{i} + (4 + 2t)\mathbf{j}$$

b) Due north: $\mathbf{i}$ component zero ($\mathbf{j}$ component positive)

$$20 - 4t = 0$$

$$t = 5$$

$$\mathbf{j} \text{ component: } 4 + 10 = 14$$

$$5 \text{ pm, } 14 \text{ km apart}$$

c) Let D be the square of the distance

$$D = (20 - 4t)^2 + (4 + 2t)^2$$

$$D = 400 - 160t + 16t^2 + 16 + 16t + 4t^2$$

$$D = 20t^2 - 144t + 416$$

Minimise D : differentiate and set to zero

$$\frac{dD}{dt} = 40t - 144 = 0$$

$$t = 3.6$$

$$\frac{d^2D}{dt^2} = 40 > 0, \text{ so a minimum}$$

$$D = 20(3.6)^2 - 144(3.6) + 416$$

$$D = 259.2 - 518.4 + 416 = 156.8$$

$$\text{distance} = \sqrt{156.8} = 12.521...$$

$$3:36 \text{ pm, } 12.5 \text{ km}$$

[In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively, and position vectors are given relative to a fixed origin O .]

At time t seconds, where $t > 0$, a particle P has velocity $\mathbf{v}$ m s⁻¹, where

$$\mathbf{v} = 6t^{\frac{1}{2}}\mathbf{i} + (2t - 5)\mathbf{j}$$

- Find the magnitude of the acceleration of P when $t = 4$
- P is at the point A when $t = 1$ and at the point B when $t = 4$. Find the exact distance AB .
- When $t = 1$, the position vector of P is $(-2\mathbf{i} + 3\mathbf{j})$ m. Find the position vector of P at time t .
- Show that P is never due east of O .

(10 marks)

a) Differentiate $\mathbf{v}$

$$\mathbf{a} = 6 \times \frac{1}{2} t^{-1/2} \mathbf{i} + 2\mathbf{j}$$

$$\mathbf{a} = 3t^{-1/2} \mathbf{i} + 2\mathbf{j}$$

$$t = 4: \mathbf{a} = \frac{3}{2} \mathbf{i} + 2\mathbf{j}$$

$$|\mathbf{a}| = \sqrt{(2.25 + 4)} = \sqrt{6.25}$$

$$\boxed{2.5 \text{ m s}^{-2}}$$

b) Displacement = $\int \mathbf{v} \, dt$ from $t = 1$ to $t = 4$

$$\int \mathbf{v} \, dt = 6 \times \frac{2}{3} t^{3/2} \mathbf{i} + (t^2 - 5t)\mathbf{j}$$

$$= 4t^{3/2} \mathbf{i} + (t^2 - 5t)\mathbf{j}$$

Limits: $[\dots]$ from 1 to 4

$$\mathbf{i}: 4(8) - 4(1) = 28$$

$$\mathbf{j}: (16 - 20) - (1 - 5) = 0$$

$$AB = 28\mathbf{i} + 0\mathbf{j}$$

$$\boxed{AB = 28 \text{ m}}$$

c) $\mathbf{r} = 4t^{3/2} \mathbf{i} + (t^2 - 5t)\mathbf{j} + \mathbf{c}$

$t = 1: \mathbf{r} = -2\mathbf{i} + 3\mathbf{j}$

$$4\mathbf{i} - 4\mathbf{j} + \mathbf{c} = -2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{c} = -6\mathbf{i} + 7\mathbf{j}$$

$$\boxed{\mathbf{r} = (4t^{3/2} - 6)\mathbf{i} + (t^2 - 5t + 7)\mathbf{j}}$$

d) Due east of O needs the $\mathbf{j}$ component to be zero

$$t^2 - 5t + 7 = 0$$

$$\text{discriminant} = 25 - 28 = -3$$

$-3 < 0$, so there are no real roots

the j component is never 0, so P is never due east of O