



1 A curve has equation $y = 4x^5 - 3x^2 + 7x - 9$

a. Find $\frac{dy}{dx}$

b. Find the gradient of the curve at the point where $x = -1$

(3 marks)

a)

Multiply by the power, then take 1 off the power

$$y = 4x^5 - 3x^2 + 7x - 9$$

$$\frac{dy}{dx} = 20x^4 - 6x + 7$$

b)

Substitute $x = -1$ into $\frac{dy}{dx}$

$$\frac{dy}{dx} = 20(-1)^4 - 6(-1) + 7$$

$$\frac{dy}{dx} = 20 + 6 + 7$$

$$\frac{dy}{dx} = 33$$

$$\boxed{\text{gradient} = 33}$$

2 $f(x) = \frac{5}{x^2} + 8\sqrt{x} - 3, \quad x > 0$

Find $f'(x)$, giving each term in its simplest form.

(3 marks)

Write each term as a power of x

$$f(x) = 5x^{-2} + 8x^{\frac{1}{2}} - 3$$

Multiply by the power, then take 1 off the power

$$f'(x) = -10x^{-3} + 4x^{-\frac{1}{2}}$$

$$f'(x) = -\frac{10}{x^3} + \frac{4}{\sqrt{x}}$$

$$\boxed{f'(x) = -\frac{10}{x^3} + \frac{4}{\sqrt{x}}}$$

3

$$y = 2x^3 - 8x^{\frac{3}{2}}, \quad x > 0$$

Find

a. $\frac{dy}{dx}$

b. $\frac{d^2y}{dx^2}$

(3 marks)

a)

$$8 \times \frac{3}{2} = 12 \text{ and } \frac{3}{2} - 1 = \frac{1}{2}$$

$$\frac{dy}{dx} = 6x^2 - 12x^{\frac{1}{2}}$$

b)

$$\text{Differentiate } \frac{dy}{dx} \text{ again: } 12 \times \frac{1}{2} = 6$$

$$\frac{d^2y}{dx^2} = 12x - 6x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 12x - \frac{6}{\sqrt{x}}$$

$$\boxed{\frac{d^2y}{dx^2} = 12x - \frac{6}{\sqrt{x}}}$$

4

$$f(x) = 2x^2 + 3x$$

Prove, from first principles, that $f'(x) = 4x + 3$

(4 marks)

$$f'(x) \text{ is the limit, as } h \rightarrow 0, \text{ of } \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 2(x+h)^2 + 3(x+h)$$

$$f(x+h) = 2x^2 + 4xh + 2h^2 + 3x + 3h$$

$$\text{Subtract } f(x) = 2x^2 + 3x$$

$$f(x+h) - f(x) = 4xh + 2h^2 + 3h$$

Divide by h

$$\frac{f(x+h) - f(x)}{h} = 4x + 2h + 3$$

$$\text{As } h \rightarrow 0, 2h \rightarrow 0$$

$$f'(x) = 4x + 3$$

$$\boxed{f'(x) = 4x + 3 \text{ as required}}$$

5

Prove, from first principles, that the derivative of x^3 is $3x^2$

(4 marks)

Let $f(x) = x^3$ and use the gradient of the chord from x to $x + h$ Expand $(x + h)^3$ using the binomial expansion

$$f(x + h) = (x + h)^3$$

$$f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3$$

$$f(x + h) - f(x) = 3x^2h + 3xh^2 + h^3$$

$$\frac{f(x + h) - f(x)}{h} = \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$\frac{f(x + h) - f(x)}{h} = 3x^2 + 3xh + h^2$$

As $h \rightarrow 0$, $3xh \rightarrow 0$ and $h^2 \rightarrow 0$

$$f'(x) = 3x^2$$

derivative of x^3 is $3x^2$

6

The curve C has equation $y = 3x^2 - x$

The point $P(1, 2)$ lies on C . The point Q also lies on C and has x coordinate $1 + h$, where $h \neq 0$

- Show that the gradient of the chord PQ is $5 + 3h$
- Explain how your answer to part (a) gives the gradient of the tangent to C at P , and state its value. (4 marks)

a)

Find the y coordinate of Q

$$y = 3(1+h)^2 - (1+h)$$

$$y = 3 + 6h + 3h^2 - 1 - h$$

$$y = 2 + 5h + 3h^2$$

$$\text{Gradient} = \frac{\text{change in } y}{\text{change in } x}$$

$$\text{gradient of } PQ = \frac{(2 + 5h + 3h^2) - 2}{(1+h) - 1}$$

$$\text{gradient of } PQ = \frac{5h + 3h^2}{h}$$

$$\text{gradient of } PQ = 5 + 3h$$

b)

As $h \rightarrow 0$, Q moves along the curve to P ,
so the chord PQ becomes the tangent at P

$$\text{gradient of tangent} = 5 + 3(0)$$

$$\boxed{\text{gradient of tangent} = 5}$$

7

$$y = \frac{x^2 + 6}{2\sqrt{x}}, \quad x > 0$$

Show that $\frac{dy}{dx} = \frac{3(x^2 - 2)}{4x\sqrt{x}}$

(4 marks)

Split the fraction and write each term as a power of x

$$y = \frac{x^2}{2x^{\frac{1}{2}}} + \frac{6}{2x^{\frac{1}{2}}}$$

$$y = \frac{1}{2}x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$$

Differentiate

$$\frac{dy}{dx} = \frac{3}{4}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$$

Write both terms over $4x^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{3x^2}{4x^{\frac{3}{2}}} - \frac{6}{4x^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{3x^2 - 6}{4x^{\frac{3}{2}}}$$

$$x^{\frac{3}{2}} = x\sqrt{x}$$

$$\frac{dy}{dx} = \frac{3(x^2 - 2)}{4x\sqrt{x}}$$

$\frac{dy}{dx} = \frac{3(x^2 - 2)}{4x\sqrt{x}} \text{ as required}$

8

A curve has equation $y = (3 - \sqrt{x})^2$, $x > 0$

a. Find $\frac{dy}{dx}$

b. Find the value of x at the point on the curve where the gradient is $\frac{1}{2}$

(5 marks)

a)

Expand the bracket first

$$y = (3 - \sqrt{x})(3 - \sqrt{x})$$

$$y = 9 - 6\sqrt{x} + x$$

$$y = 9 - 6x^{\frac{1}{2}} + x$$

$$\frac{dy}{dx} = -3x^{-\frac{1}{2}} + 1$$

$$\frac{dy}{dx} = 1 - \frac{3}{\sqrt{x}}$$

b)

$$1 - \frac{3}{\sqrt{x}} = \frac{1}{2}$$

$$\frac{3}{\sqrt{x}} = \frac{1}{2}$$

$$\sqrt{x} = 6$$

$$x = 36$$

9

 $f(x) = ax^3 + bx^2 - 3x$, where a and b are constants.Given that $f'(2) = 17$ and $f''(2) = 22$, find the value of a and the value of b .

(5 marks)

Differentiate twice

$$f'(x) = 3ax^2 + 2bx - 3$$

$$f''(x) = 6ax + 2b$$

Use $f'(2) = 17$

$$12a + 4b - 3 = 17$$

$$12a + 4b = 20$$

$$3a + b = 5$$

Use $f''(2) = 22$

$$12a + 2b = 22$$

$$6a + b = 11$$

Subtract the two equations

$$3a = 6$$

$$a = 2$$

$$b = 5 - 3(2)$$

$$b = -1$$

$$a = 2, b = -1$$

10

The curve C has equation $y = \frac{16}{x} + kx^{\frac{3}{2}}$, $x > 0$, where k is a constant.

The gradient of C at the point where $x = 4$ is 5

a. Find the value of k .

b. Find the value of $\frac{d^2y}{dx^2}$ at the point where $x = 4$

(6 marks)

a)

Write as powers of x

$$y = 16x^{-1} + kx^{\frac{3}{2}}$$

$$\frac{dy}{dx} = -16x^{-2} + \frac{3k}{2}x^{\frac{1}{2}}$$

Gradient is 5 when $x = 4$

$$-\frac{16}{16} + \frac{3k}{2}(2) = 5$$

$$-1 + 3k = 5$$

$$3k = 6$$

$$k = 2$$

b)

With $k = 2$, $\frac{dy}{dx} = -16x^{-2} + 3x^{\frac{1}{2}}$

$$\frac{d^2y}{dx^2} = 32x^{-3} + \frac{3}{2}x^{-\frac{1}{2}}$$

Substitute $x = 4$

$$\frac{d^2y}{dx^2} = \frac{32}{64} + \frac{3}{2} \times \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{3}{4}$$

$$\frac{d^2y}{dx^2} = \frac{5}{4}$$

11

Water drains out of a tank. The depth of water, h cm, t minutes after the water starts to drain is modelled by

$$h = 64 - 16\sqrt{t} + t, \quad 0 \leq t \leq 64$$

a. Find $\frac{dh}{dt}$

b. Find the rate at which the depth is changing when $t = 16$, stating whether the depth is rising or falling.

c. Find the time at which the depth is falling at 3 cm per minute.

(6 marks)

a)

$$h = 64 - 16t^{\frac{1}{2}} + t$$

$$\frac{dh}{dt} = -8t^{-\frac{1}{2}} + 1$$

$$\frac{dh}{dt} = 1 - \frac{8}{\sqrt{t}}$$

b)

$$\frac{dh}{dt} = 1 - \frac{8}{\sqrt{16}}$$

$$\frac{dh}{dt} = 1 - 2$$

$$\frac{dh}{dt} = -1$$

Negative, so the depth is going down

falling at 1 cm per minute

c)

Falling at 3 cm per minute means $\frac{dh}{dt} = -3$

$$1 - \frac{8}{\sqrt{t}} = -3$$

$$\frac{8}{\sqrt{t}} = 4$$

$$\sqrt{t} = 2$$

$t = 4$ minutes

12

$f(x) = ax^2 + bx + c$, where a , b and c are constants.

a. Prove, from first principles, that $f'(x) = 2ax + b$

The curve with equation $y = f(x)$

- passes through the point $(0, 5)$
- has gradient 1 at the point where $x = 1$

and $f''(x) = 6$

b. Find $f(x)$.

(7 marks)

a)

$$f(x+h) = a(x+h)^2 + b(x+h) + c$$

$$f(x+h) = ax^2 + 2axh + ah^2 + bx + bh + c$$

$$f(x+h) - f(x) = 2axh + ah^2 + bh$$

$$\frac{f(x+h) - f(x)}{h} = 2ax + ah + b$$

As $h \rightarrow 0$, $ah \rightarrow 0$

$$f'(x) = 2ax + b$$

b)

Differentiate again: $f''(x) = 2a$

$$2a = 6$$

$$a = 3$$

Gradient 1 at $x = 1$

$$f'(1) = 2(3)(1) + b$$

$$6 + b = 1$$

$$b = -5$$

Passes through $(0, 5)$

$$f(0) = c$$

$$c = 5$$

$$f(x) = 3x^2 - 5x + 5$$