



- 1** Find  $\int (6x^2 - 4x + 5) dx$ , simplifying each term.

(3 marks)

Add 1 to the power, divide by the new power

$$\int (6x^2 - 4x + 5) dx = \frac{6x^3}{3} - \frac{4x^2}{2} + 5x + c$$

$$\int (6x^2 - 4x + 5) dx = 2x^3 - 2x^2 + 5x + c$$

Do not forget + c

$$\boxed{2x^3 - 2x^2 + 5x + c}$$

- 2** Find  $\int (3\sqrt{x} + \frac{4}{x^3}) dx$ , giving each term in its simplest form.

(4 marks)

Write each term as a power of x

$$= \int (3x^{1/2} + 4x^{-3}) dx$$

Add 1 to each power

$$= \frac{3x^{3/2}}{(3/2)} + \frac{4x^{-2}}{(-2)} + c$$

$$= 2x^{3/2} - 2x^{-2} + c$$

$$\boxed{2x^{3/2} - \frac{2}{x^2} + c}$$

3 A curve with equation  $y = f(x)$  passes through the point (2, 9)

Given that  $f'(x) = 4x^3 - 6x + 1$ , find  $f(x)$

(4 marks)

Integrate  $f'(x)$

$$f(x) = \frac{4x^4}{4} - \frac{6x^2}{2} + x + c$$

$$f(x) = x^4 - 3x^2 + x + c$$

Use the point (2, 9) to find  $c$

$$9 = 2^4 - 3(2)^2 + 2 + c$$

$$9 = 16 - 12 + 2 + c$$

$$9 = 6 + c$$

$$c = 3$$

$$f(x) = x^4 - 3x^2 + x + 3$$

4

Use algebraic integration to find the exact value of  $\int_1^4 (2x - \frac{3}{\sqrt{x}}) dx$

Solutions relying entirely on calculator technology are not acceptable.

(4 marks)

Write the root as a power

$$= 2x - 3x^{-1/2}$$

Integrate

$$\begin{aligned} \int (2x - 3x^{-1/2}) dx &= x^2 - \frac{3x^{1/2}}{(1/2)} \\ &= x^2 - 6x^{1/2} \end{aligned}$$

Substitute the limits:  $[x^2 - 6\sqrt{x}]$  from 1 to 4

$$= (4^2 - 6\sqrt{4}) - (1^2 - 6\sqrt{1})$$

$$= (16 - 12) - (1 - 6)$$

$$= 4 + 5$$

$$= 9$$

$$9$$

5

The curve  $C$  has equation  $y = f(x)$ ,  $x > 0$ , and passes through the point  $(1, 2)$

Given that  $f'(x) = \frac{(x^2 + 2)^2}{x^2}$

- a. show that  $f'(x) = x^2 + A + Bx^{-2}$  where  $A$  and  $B$  are constants to be found  
 b. find  $f(x)$ , simplifying each term.

(6 marks)

a)

Expand the bracket

$$(x^2 + 2)^2 = x^4 + 4x^2 + 4$$

Divide every term by  $x^2$

$$f'(x) = \frac{x^4}{x^2} + \frac{4x^2}{x^2} + \frac{4}{x^2}$$

$$f'(x) = x^2 + 4 + 4x^{-2}$$

$$A = 4, B = 4$$

b)

$$f(x) = \frac{x^3}{3} + 4x + \frac{4x^{-1}}{(-1)} + c$$

$$f(x) = \frac{1}{3}x^3 + 4x - 4x^{-1} + c$$

Use  $(1, 2)$

$$2 = \frac{1}{3} + 4 - 4 + c$$

$$c = \frac{5}{3}$$

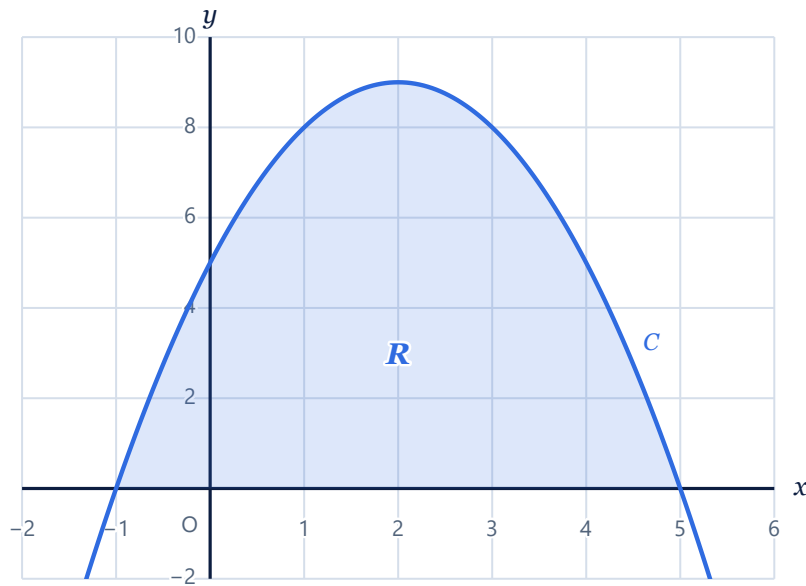
$$f(x) = \frac{1}{3}x^3 + 4x - \frac{4}{x} + \frac{5}{3}$$

6

The curve  $C$  has equation  $y = 5 + 4x - x^2$

The finite region  $R$  is bounded by  $C$  and the  $x$ -axis.

Use algebraic integration to find the area of  $R$



(5 marks)

Find where  $C$  meets the  $x$ -axis

$$5 + 4x - x^2 = 0$$

$$(5 - x)(1 + x) = 0$$

$$x = -1 \text{ or } x = 5$$

Integrate between the roots

$$\int (5 + 4x - x^2) dx = 5x + 2x^2 - \frac{x^3}{3}$$

$$\left[ 5x + 2x^2 - \frac{x^3}{3} \right] \text{ from } -1 \text{ to } 5$$

$$= \left( 25 + 50 - \frac{125}{3} \right) - \left( -5 + 2 + \frac{1}{3} \right)$$

$$= \frac{100}{3} - \left( -\frac{8}{3} \right)$$

$$= \frac{108}{3}$$

$$= 36$$

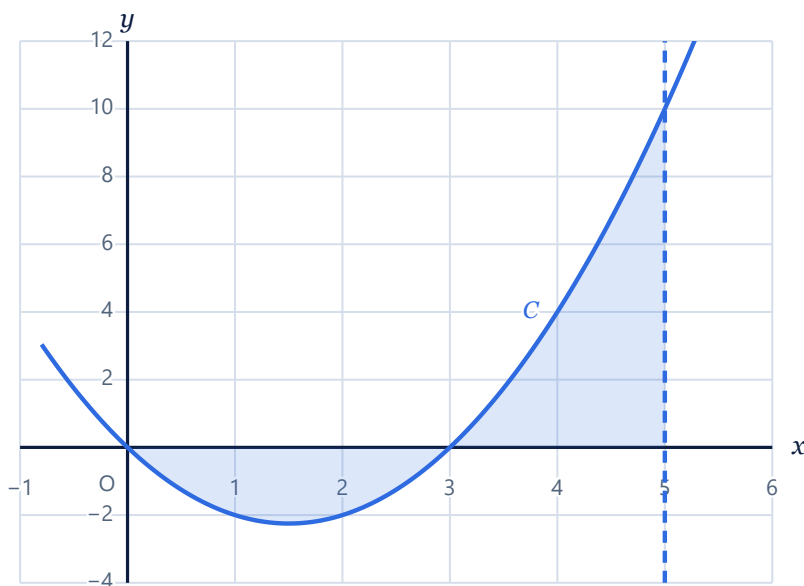
$$\boxed{\text{area of } R = 36}$$

7

The curve  $C$  has equation  $y = x^2 - 3x$

a. Find the total area of the regions bounded by  $C$ , the  $x$ -axis and the lines  $x = 0$  and  $x = 5$

b. Find the value of  $\int_0^5 (x^2 - 3x) dx$  and explain why it is not the same as your answer to part (a).



(6 marks)

a)

Find where  $C$  crosses the  $x$ -axis

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$$x = 0 \text{ or } x = 3$$

below the axis for  $0 < x < 3$ , above for  $3 < x < 5$

$$\int (x^2 - 3x) dx = \frac{x^3}{3} - \frac{3x^2}{2}$$

[ ] from 0 to 3

$$= \left(9 - \frac{27}{2}\right) - 0$$

$$= -\frac{9}{2}$$

so that area is  $\frac{9}{2}$

[ ] from 3 to 5

$$= \left(\frac{125}{3} - \frac{75}{2}\right) - \left(-\frac{9}{2}\right)$$

$$= \frac{25}{6} + \frac{27}{6}$$

$$= \frac{26}{3}$$

Add the two areas

$$\text{total} = \frac{9}{2} + \frac{26}{3}$$

$$\text{total} = \frac{27}{6} + \frac{52}{6}$$

$$\text{total} = \frac{79}{6}$$

$$\boxed{\frac{79}{6}}$$

b)

$$\begin{aligned} \int_{\text{from } 0 \text{ to } 5} &= -\frac{9}{2} + \frac{26}{3} \\ &= \frac{25}{6} \end{aligned}$$

the integral counts area below the x-axis as negative

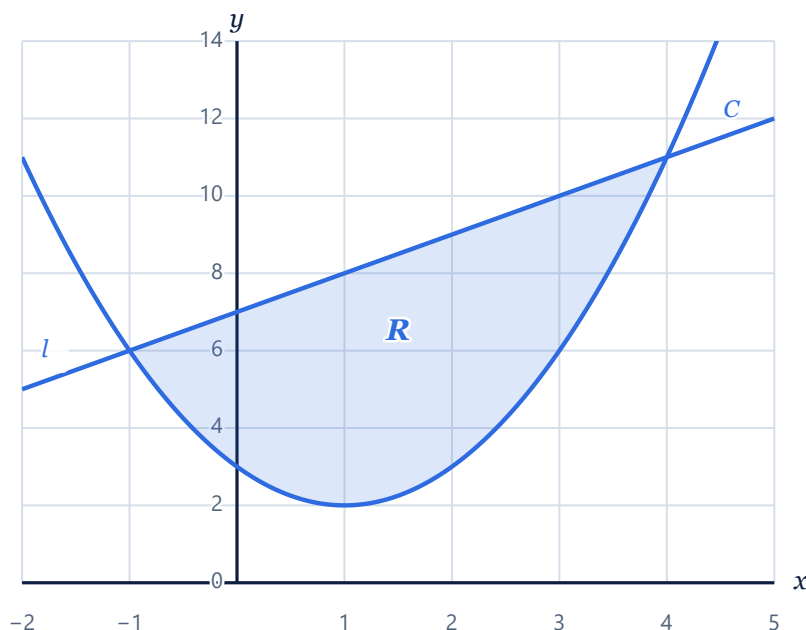
$$\boxed{\frac{25}{6}}$$

8

The curve  $C$  has equation  $y = x^2 - 2x + 3$  and the line  $l$  has equation  $y = x + 7$

a. Find the coordinates of the points where  $l$  meets  $C$

b. The finite region  $R$  is bounded by  $C$  and  $l$ . Use algebraic integration to find the exact area of  $R$



(7 marks)

a)

Solve the equations together

$$x^2 - 2x + 3 = x + 7$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x = -1: y = 6$$

$$x = 4: y = 11$$

$$\boxed{(-1, 6) \text{ and } (4, 11)}$$

b)

Area =  $\int (\text{top} - \text{bottom}) dx$ , the line is on top

$$\text{top} - \text{bottom} = (x + 7) - (x^2 - 2x + 3)$$

$$\text{top} - \text{bottom} = -x^2 + 3x + 4$$

$$\int (-x^2 + 3x + 4) dx = -\frac{x^3}{3} + \frac{3x^2}{2} + 4x$$

[ ] from -1 to 4

$$= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)$$

$$= \frac{56}{3} - \left(-\frac{13}{6}\right)$$

$$= \frac{112}{6} + \frac{13}{6}$$

$$= \frac{125}{6}$$

|                 |
|-----------------|
| $\frac{125}{6}$ |
|-----------------|

9

Given that  $k > 2$  and  $\int_2^k (3x^2 - 4) dx = 48$

a. show that  $k^3 - 4k - 48 = 0$

b. Hence show that  $k = 4$  is the only solution.

(5 marks)

a)

$$\int (3x^2 - 4) dx = x^3 - 4x$$

$[x^3 - 4x]$  from 2 to  $k$

$$= (k^3 - 4k) - (8 - 8)$$

$$= k^3 - 4k$$

$$k^3 - 4k = 48$$

$$k^3 - 4k - 48 = 0$$

b)

Factor theorem: try  $k = 4$

$$4^3 - 4(4) - 48 = 64 - 16 - 48 = 0$$

so  $(k - 4)$  is a factor

$$k^3 - 4k - 48 = (k - 4)(k^2 + 4k + 12)$$

Discriminant of the quadratic

$$b^2 - 4ac = 16 - 48 = -32$$

$-32 < 0$  so  $k^2 + 4k + 12 = 0$  has no real roots

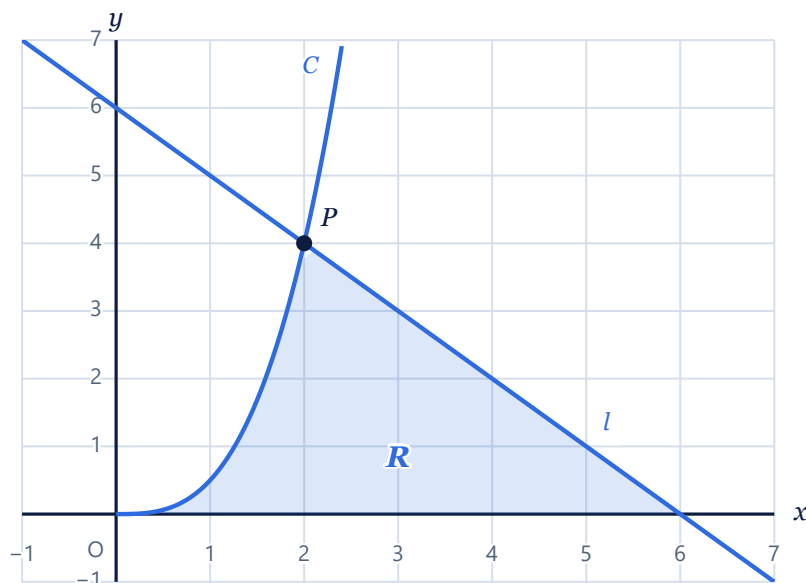
|                              |
|------------------------------|
| $k = 4$ is the only solution |
|------------------------------|

10

The curve  $C$  has equation  $y = \frac{1}{2}x^3$ ,  $x \geq 0$ , and the line  $l$  has equation  $y = 6 - x$

- Show that  $l$  meets  $C$  at exactly one point,  $P(2, 4)$
- The region  $R$  is bounded by  $C$ ,  $l$  and the  $x$ -axis. Find the area of  $R$

Solutions relying entirely on calculator technology are not acceptable.



(7 marks)

a)

$$\frac{1}{2}x^3 = 6 - x$$

$$x^3 = 12 - 2x$$

$$x^3 + 2x - 12 = 0$$

$$x = 2 \text{ is a root: } 8 + 4 - 12 = 0$$

$$(x - 2)(x^2 + 2x + 6) = 0$$

$$b^2 - 4ac = 4 - 24 = -20 < 0$$

so the quadratic has no real roots

$$x = 2: y = 6 - 2 = 4$$

meets only at  $P(2, 4)$

b)

Split  $R$  at  $x = 2$ : under the curve, then a triangle under the line

$$\int \frac{1}{2}x^3 dx = \frac{x^4}{8}$$

$$\left[ \frac{x^4}{8} \right] \text{ from } 0 \text{ to } 2$$

$$= \frac{16}{8} - 0 = 2$$

$L$  meets the  $x$ -axis at  $x = 6$

$$\text{triangle} = \frac{1}{2} \times (6 - 2) \times 4$$

$$\text{triangle} = 8$$

$$\text{area of } R = 2 + 8$$

$$\boxed{10}$$

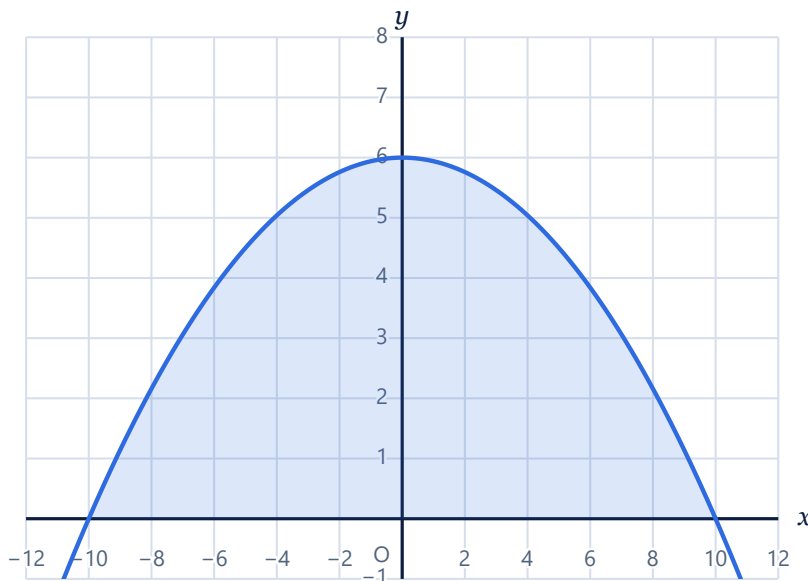
11

The entrance to a road tunnel is modelled by the region between the ground and the curve with equation

$$y = 6 - 0.06x^2$$

where  $x$  metres is the horizontal distance from the centre of the entrance and  $y$  metres is the height above the ground.

- Find the width of the tunnel at ground level.
- Use algebraic integration to find the area of the entrance.
- The tunnel is 50 m long and has the same cross-section all the way through. Find the volume of the tunnel.
- Give one reason why the model may not give an accurate answer for the amount of space available for traffic.



(8 marks)

a)

Ground level means  $y = 0$

$$6 - 0.06x^2 = 0$$

$$x^2 = \frac{6}{0.06}$$

$$x^2 = 100$$

$$x = \pm 10$$

$$\text{width} = 10 - (-10)$$

20 m

b)

$$\int (6 - 0.06x^2) dx = 6x - 0.02x^3$$

$$[6x - 0.02x^3] \text{ from } -10 \text{ to } 10$$

$$= (60 - 20) - (-60 + 20)$$

$$= 40 + 40$$

$$= 80$$

$$80 \text{ m}^2$$

c)

Volume = cross-section area  $\times$  length

$$V = 80 \times 50$$

$$4000 \text{ m}^3$$

d)

the road surface and fittings take up some of the space

12

The curve  $C$  has equation  $y = x(k - x)$ , where  $k$  is a positive constant.

- a. Show that the area of the finite region bounded by  $C$  and the  $x$ -axis is  $\frac{k^3}{6}$

The line  $l$  with equation  $y = 2x$  meets  $C$  at the origin  $O$  and at the point  $P$ .

- b. Find the  $x$ -coordinate of  $P$  in terms of  $k$   
c. Given that the finite region bounded by  $C$  and  $l$  has area 36, find the value of  $k$

(9 marks)

a)

$C$  meets the  $x$ -axis where  $x(k - x) = 0$

$$x = 0 \text{ or } x = k$$

$$\int (kx - x^2) dx = \frac{kx^2}{2} - \frac{x^3}{3}$$

[ ] from 0 to  $k$

$$= \frac{k^3}{2} - \frac{k^3}{3}$$

$$= \frac{3k^3}{6} - \frac{2k^3}{6}$$

$$= \frac{k^3}{6}$$

$$\boxed{\text{area} = \frac{k^3}{6}}$$

b)

$$x(k - x) = 2x$$

$$kx - x^2 - 2x = 0$$

$$x(k - 2 - x) = 0$$

$$\boxed{x = k - 2}$$

c)

Area =  $\int (\text{curve} - \text{line}) dx$  from 0 to  $k - 2$

$$\text{curve} - \text{line} = kx - x^2 - 2x$$

$$\text{curve} - \text{line} = (k - 2)x - x^2$$

Same shape as part a) with  $k - 2$  in place of  $k$

$$\text{area} = \frac{(k - 2)^3}{6}$$

$$\frac{(k - 2)^3}{6} = 36$$

$$(k - 2)^3 = 216$$

$$k - 2 = 6$$

$$k = 8$$

$$k = 8$$