



- 1 Prove that the sum of any three consecutive integers is a multiple of 3

(2 marks)

Let n be any integer

Three consecutive integers: $n - 1, n, n + 1$

$$\begin{aligned}\text{sum} &= (n - 1) + n + (n + 1) \\ &= 3n\end{aligned}$$

n is an integer, so $3n$ is a multiple of 3

The sum is always a multiple of 3

- 2 Jamal says

"If x and y are real numbers with $x > y$, then $x^2 > y^2$ "

Use a counterexample to show that Jamal is wrong.

(2 marks)

Try a negative y

$$x = 1, y = -2$$

$$1 > -2, \text{ so } x > y$$

$$x^2 = 1$$

$$y^2 = (-2)^2 = 4$$

$$1 < 4, \text{ so } x^2 \text{ is not greater than } y^2$$

$x = 1, y = -2$ is a counterexample, so Jamal is wrong

3

Prove by exhaustion that every prime number between 30 and 50 is either one more or one less than a multiple of 6

(3 marks)

List every prime between 30 and 50

31, 37, 41, 43, 47

Check each one

$$31 = 6 \times 5 + 1$$

$$37 = 6 \times 6 + 1$$

$$41 = 6 \times 7 - 1$$

$$43 = 6 \times 7 + 1$$

$$47 = 6 \times 8 - 1$$

All cases checked

Every prime between 30 and 50 is one more or one less than a multiple of 6

4

Prove that the difference between the squares of any two consecutive odd numbers is a multiple of 8

(3 marks)

Let n be an integer

Consecutive odd numbers: $2n + 1$ and $2n + 3$

$$\begin{aligned}(2n + 3)^2 - (2n + 1)^2 &= (4n^2 + 12n + 9) - (4n^2 + 4n + 1) \\ &= 8n + 8 \\ &= 8(n + 1)\end{aligned}$$

$n + 1$ is an integer

The difference is a multiple of 8

5

Prove that $x^2 - 6x + 11 > 0$ for all real values of x

(3 marks)

Complete the square

$$\begin{aligned}x^2 - 6x + 11 &= (x - 3)^2 - 9 + 11 \\ &= (x - 3)^2 + 2\end{aligned}$$

A square is never negative

$$(x - 3)^2 \geq 0$$

$$(x - 3)^2 + 2 \geq 2$$

So $x^2 - 6x + 11 \geq 2 > 0$ for all real x

6 Prove that $n^2 + 5n + 3$ is odd for every integer n

(4 marks)

Case 1: n even, $n = 2k$

$$\begin{aligned}n^2 + 5n + 3 &= (2k)^2 + 5(2k) + 3 \\&= 4k^2 + 10k + 3 \\&= 2(2k^2 + 5k + 1) + 1\end{aligned}$$

2 x integer + 1, so odd

Case 2: n odd, $n = 2k + 1$

$$\begin{aligned}n^2 + 5n + 3 &= (2k + 1)^2 + 5(2k + 1) + 3 \\&= 4k^2 + 4k + 1 + 10k + 5 + 3 \\&= 4k^2 + 14k + 9 \\&= 2(2k^2 + 7k + 4) + 1\end{aligned}$$

2 x integer + 1, so odd

Every integer is even or odd

$n^2 + 5n + 3$ is odd for every integer n

7 For each statement, either prove that it is true or use a counterexample to show that it is false.

a. The sum of any two consecutive square numbers is odd.

b. $n^2 - n + 11$ is prime for every positive integer n

(5 marks)

a) Let n be an integer

$$\begin{aligned}n^2 + (n + 1)^2 &= n^2 + n^2 + 2n + 1 \\&= 2n^2 + 2n + 1 \\&= 2(n^2 + n) + 1\end{aligned}$$

2 x integer + 1, so odd

True

b) Try $n = 11$ so every term has a factor 11

$$11^2 - 11 + 11 = 121$$

$$121 = 11 \times 11$$

121 is not prime

False: $n = 11$ is a counterexample

8

Ava is asked to prove that $n^3 - n$ is a multiple of 6 for every positive integer n . She writes "n = 1 gives 0, n = 2 gives 6, n = 3 gives 24 and n = 4 gives 60. These are all multiples of 6, so the statement is true."

- Explain why Ava has not proved the statement.
- Prove that $n^3 - n$ is a multiple of 6 for every positive integer n

(4 marks)

a)

Only 4 values checked - it must work for every positive integer, and there are infinitely many

b) Factorise fully

$$\begin{aligned} n^3 - n &= n(n^2 - 1) \\ &= (n - 1)n(n + 1) \end{aligned}$$

Three consecutive integers

at least one of them is even, so 2 is a factor

one of them is a multiple of 3, so 3 is a factor

2 and 3 are both factors, so $2 \times 3 = 6$ is a factor

$n^3 - n$ is a multiple of 6

9

The points A , B and C have coordinates $(1, 2)$, $(5, 4)$ and $(3, 8)$ respectively.

Prove that triangle ABC is a right-angled isosceles triangle.

(4 marks)

Gradients of two sides

$$\text{gradient } AB = \frac{4 - 2}{5 - 1} = \frac{1}{2}$$

$$\text{gradient } BC = \frac{8 - 4}{3 - 5} = -2$$

$$\frac{1}{2} \times -2 = -1$$

so $AB \perp BC$ and angle $ABC = 90^\circ$

Lengths of the two sides

$$AB = \sqrt{(4^2 + 2^2)} = \sqrt{20}$$

$$BC = \sqrt{((-2)^2 + 4^2)} = \sqrt{20}$$

$$AB = BC$$

Right angle at B and $AB = BC$, so ABC is right-angled and isosceles

10

a. Prove that $x^2 + y^2 \geq 2xy$ for all real numbers x and y b. Hence prove that, for all positive real numbers a and b , $\frac{a+b}{2} \geq \sqrt{ab}$

(5 marks)

a) A square is never negative

$$(x - y)^2 \geq 0$$

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy$$

$$x^2 + y^2 \geq 2xy \text{ for all real } x, y$$

b) $a, b > 0$ so $\sqrt{a}$ and $\sqrt{b}$ are realPut $x = \sqrt{a}$, $y = \sqrt{b}$ in a)

$$(\sqrt{a})^2 + (\sqrt{b})^2 \geq 2\sqrt{a}\sqrt{b}$$

$$a + b \geq 2\sqrt{ab}$$

Divide both sides by 2

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab} \text{ for all positive } a, b$$

11

Prove that one more than the product of any four consecutive integers is a perfect square.

(4 marks)

Let n be an integerFour consecutive integers: $n, n+1, n+2, n+3$

Pair the outside two and the inside two

$$n(n+3) = n^2 + 3n$$

$$(n+1)(n+2) = n^2 + 3n + 2$$

Let $m = n^2 + 3n$

$$n(n+1)(n+2)(n+3) + 1 = m(m+2) + 1$$

$$= m^2 + 2m + 1$$

$$= (m+1)^2$$

$$= (n^2 + 3n + 1)^2$$

 $n^2 + 3n + 1$ is an integer

One more than the product is a perfect square

12

Prove that the cube of any integer is either a multiple of 9, one more than a multiple of 9 or one less than a multiple of 9

(5 marks)

Every integer is $3k$, $3k + 1$ or $3k - 1$

Case 1: $n = 3k$

$$\begin{aligned}n^3 &= 27k^3 \\&= 9(3k^3)\end{aligned}$$

a multiple of 9

Case 2: $n = 3k + 1$, use the binomial expansion

$$\begin{aligned}n^3 &= 27k^3 + 27k^2 + 9k + 1 \\&= 9(3k^3 + 3k^2 + k) + 1\end{aligned}$$

one more than a multiple of 9

Case 3: $n = 3k - 1$

$$\begin{aligned}n^3 &= 27k^3 - 27k^2 + 9k - 1 \\&= 9(3k^3 - 3k^2 + k) - 1\end{aligned}$$

one less than a multiple of 9

All integers covered

Every cube is $9m$, $9m + 1$ or $9m - 1$
