



1 The curve C has equation $y = x^3 - 4x + 1$

Find an equation of the tangent to C at the point where $x = 2$, giving your answer in the form $y = mx + c$

(4 marks)

Find the point

$$y = 2^3 - 4(2) + 1$$

$$y = 1$$

Differentiate for the gradient

$$\frac{dy}{dx} = 3x^2 - 4$$

$$\frac{dy}{dx} = 3(2)^2 - 4 = 8$$

Use $y - y_1 = m(x - x_1)$

$$y - 1 = 8(x - 2)$$

$$y - 1 = 8x - 16$$

$$y = 8x - 15$$

$$\boxed{y = 8x - 15}$$

- 2 The point $P(3, 6)$ lies on the curve with equation $y = 2x^2 - 5x + 3$

Find an equation of the normal to the curve at P , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers.

(4 marks)

Gradient of the tangent at P

$$\frac{dy}{dx} = 4x - 5$$

$$\frac{dy}{dx} = 4(3) - 5 = 7$$

Normal gradient = $-1 \div$ tangent gradient

$$m = -\frac{1}{7}$$

$$y - 6 = -\frac{1}{7}(x - 3)$$

Multiply by 7

$$7y - 42 = -x + 3$$

$$x + 7y - 45 = 0$$

$$\boxed{x + 7y - 45 = 0}$$

- 3 $f(x) = x^3 + 3x^2 - 24x + 5$

Find the set of values of x for which $f(x)$ is a decreasing function.

(4 marks)

Decreasing means $f'(x) < 0$

$$f'(x) = 3x^2 + 6x - 24$$

$$f'(x) = 3(x^2 + 2x - 8)$$

$$f'(x) = 3(x + 4)(x - 2)$$

Critical values

$$x = -4 \text{ or } x = 2$$

Positive x^2 coefficient, so the graph of $f'(x)$ is a U shape: negative between the roots

$$= 3(x + 4)(x - 2) < 0$$

$$\boxed{-4 < x < 2}$$

4

The curve C has equation $y = x^3 - 6x^2 + 9x + 2$

- Find the coordinates of the stationary points of C
- Determine the nature of each stationary point, giving a reason for your answer.

(6 marks)

a)

Stationary points: $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$0 = 3(x^2 - 4x + 3)$$

$$0 = 3(x - 1)(x - 3)$$

$$x = 1 \text{ or } x = 3$$

$$x = 1: y = 1 - 6 + 9 + 2 = 6$$

$$x = 3: y = 27 - 54 + 27 + 2 = 2$$

$(1, 6)$ and $(3, 2)$

b)

Second derivative test

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$x = 1: \frac{d^2y}{dx^2} = -6 < 0$$

so $(1, 6)$ is a maximum

$$x = 3: \frac{d^2y}{dx^2} = 6 > 0$$

so $(3, 2)$ is a minimum

$(1, 6)$ maximum, $(3, 2)$ minimum

5

A curve has equation $y = x^2 + \frac{16}{x}$, $x > 0$

a. Find (i) $\frac{dy}{dx}$ (ii) $\frac{d^2y}{dx^2}$

b. Verify that the curve has a stationary point when $x = 2$

c. Determine the nature of this stationary point, giving a reason for your answer.

(7 marks)

a)

Write as powers of x

$$y = x^2 + 16x^{-1}$$

$$\frac{dy}{dx} = 2x - 16x^{-2}$$

$$\frac{d^2y}{dx^2} = 2 + 32x^{-3}$$

b)

Substitute $x = 2$ into $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2(2) - \frac{16}{2^2}$$

$$\frac{dy}{dx} = 4 - 4$$

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0 \text{ at } x = 2, \text{ so it is a stationary point}$$

c)

$$\frac{d^2y}{dx^2} = 2 + \frac{32}{2^3}$$

$$\frac{d^2y}{dx^2} = 2 + 4 = 6$$

$$6 > 0$$

minimum point

6

$$f(x) = x^3 - 3x^2 + 5x - 7$$

Prove that $f(x)$ is an increasing function for all real values of x

(4 marks)

Increasing means $f'(x) > 0$

$$f'(x) = 3x^2 - 6x + 5$$

Complete the square

$$f'(x) = 3(x^2 - 2x) + 5$$

$$f'(x) = 3[(x - 1)^2 - 1] + 5$$

$$f'(x) = 3(x - 1)^2 + 2$$

A square is never negative

$$= (x - 1)^2 \geq 0$$

$$= 3(x - 1)^2 + 2 \geq 2$$

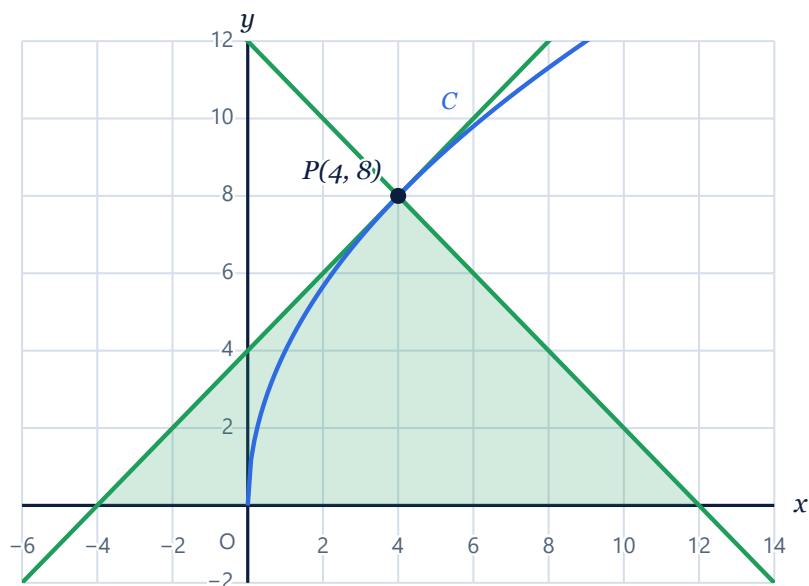
so $f'(x) > 0$ for all x

$f(x)$ is increasing for all real x

7

The curve C has equation $y = 4\sqrt{x}$, $x > 0$. The point $P(4, 8)$ lies on C .

- Find an equation of the tangent to C at P
- Find an equation of the normal to C at P
- The tangent and the normal meet the x -axis at A and B . Find the area of triangle APB



(7 marks)

a)

Write the root as a power

$$y = 4x^{1/2}$$

$$\frac{dy}{dx} = 2x^{-1/2}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4}} = 1$$

$$y - 8 = 1(x - 4)$$

$$y = x + 4$$

$$\boxed{y = x + 4}$$

b)

Normal gradient = $-1 \div 1$

$$m = -1$$

$$y - 8 = -1(x - 4)$$

$$y = -x + 12$$

$$\boxed{y = -x + 12}$$

c)

Put $y = 0$ in each line

$$0 = x + 4 \text{ so } A = (-4, 0)$$

$$0 = -x + 12 \text{ so } B = (12, 0)$$

$$\text{base } AB = 12 - (-4) = 16$$

$$\text{height} = y\text{-coordinate of } P = 8$$

$$\text{area} = \frac{1}{2} \times 16 \times 8$$

$$\text{area} = 64$$

64

8

The curve C has equation $y = 2x^3 - 9x^2 + 12x - 1$

Find equations of the two tangents to C that are parallel to the line $y = 12x$

(5 marks)

Parallel lines have equal gradients

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

$$6x^2 - 18x + 12 = 12$$

$$6x^2 - 18x = 0$$

$$6x(x - 3) = 0$$

$$x = 0 \text{ or } x = 3$$

Find the y-coordinates

$$x = 0: y = -1$$

$$x = 3: y = 54 - 81 + 36 - 1 = 8$$

Tangent at $(0, -1)$

$$y = 12x - 1$$

Tangent at $(3, 8)$

$$y - 8 = 12(x - 3)$$

$$y = 12x - 28$$

$$y = 12x - 1 \text{ and } y = 12x - 28$$

9

The curve with equation $y = ax^3 + bx^2 - 4$, where a and b are constants, has a stationary point at $(2, -12)$

- Find the value of a and the value of b
- Find the coordinates of the other stationary point and determine the nature of both stationary points.

(7 marks)

a)

Stationary at $x = 2$ means $\frac{dy}{dx} = 0$ there

$$\frac{dy}{dx} = 3ax^2 + 2bx$$

$$12a + 4b = 0$$

$$b = -3a$$

$(2, -12)$ Lies on the curve

$$8a + 4b - 4 = -12$$

$$8a + 4b = -8$$

$$8a - 12a = -8$$

$$-4a = -8$$

$$a = 2$$

$$b = -6$$

$$a = 2, b = -6$$

b)

$$y = 2x^3 - 6x^2 - 4$$

$$\frac{dy}{dx} = 6x^2 - 12x$$

$$0 = 6x(x - 2)$$

$$x = 0: y = -4$$

Second derivative test

$$\frac{d^2y}{dx^2} = 12x - 12$$

$$x = 0: \frac{d^2y}{dx^2} = -12 < 0$$

so $(0, -4)$ is a maximum

$$x = 2: \frac{d^2y}{dx^2} = 12 > 0$$

so $(2, -12)$ is a minimum

$(0, -4)$ maximum, $(2, -12)$ minimum

10

A closed gift box is a cuboid with a square base of side x cm and a height of h cm. The volume of the box is 500 cm^3 .

- Show that the surface area, $A \text{ cm}^2$, of the box is given by $A = 2x^2 + \frac{2000}{x}$
- Use calculus to find the minimum value of A , giving your answer to 3 significant figures.
- Justify that the value you found is a minimum.

(9 marks)

a)

Use the volume to write h in terms of x

$$x^2 h = 500$$

$$h = \frac{500}{x^2}$$

Top and bottom, plus four sides

$$A = 2x^2 + 4xh$$

$$A = 2x^2 + 4x \times \frac{500}{x^2}$$

$$A = 2x^2 + \frac{2000}{x}$$

$$A = 2x^2 + \frac{2000}{x}$$

b)

$$A = 2x^2 + 2000x^{-1}$$

$$\frac{dA}{dx} = 4x - 2000x^{-2}$$

$$\text{Set } \frac{dA}{dx} = 0$$

$$4x = \frac{2000}{x^2}$$

$$x^3 = 500$$

$$x = \sqrt[3]{500} = 7.937...$$

$$A = 2(7.937...) ^2 + \frac{2000}{7.937...}$$

$$A = 125.99... + 251.98...$$

$$A = 377.97...$$

$$378 \text{ cm}^2$$

c)

$$\frac{d^2A}{dx^2} = 4 + 4000x^{-3}$$

$$\frac{d^2A}{dx^2} = 4 + \frac{4000}{500} = 12$$

$$12 > 0$$

minimum

11

An open box is made from a rectangular sheet of metal 24 cm by 15 cm. A square of side x cm is cut from each corner and the sides are folded up.

- Show that the volume, $V \text{ cm}^3$, of the box is $V = 4x^3 - 78x^2 + 360x$
- State the range of possible values of x
- Use calculus to find the maximum volume of the box, justifying that it is a maximum.

(8 marks)

a)

Length, width and height of the box

$$V = x(24 - 2x)(15 - 2x)$$

$$V = x(360 - 48x - 30x + 4x^2)$$

$$V = x(4x^2 - 78x + 360)$$

$$V = 4x^3 - 78x^2 + 360x$$

$$V = 4x^3 - 78x^2 + 360x$$

b)

Every length must be positive

$$= 15 - 2x > 0$$

$$= x < 7.5$$

$$0 < x < 7.5$$

c)

$$\frac{dV}{dx} = 12x^2 - 156x + 360$$

$$0 = 12(x^2 - 13x + 30)$$

$$0 = 12(x - 3)(x - 10)$$

$$x = 3 \text{ or } x = 10$$

$x = 10$ is not in $0 < x < 7.5$, so $x = 3$

$$V = 3 \times 18 \times 9$$

$$V = 486$$

Check it is a maximum

$$\frac{d^2V}{dx^2} = 24x - 156$$

$$\frac{d^2V}{dx^2} = 72 - 156 = -84 < 0$$

$$\text{maximum volume } 486 \text{ cm}^3$$

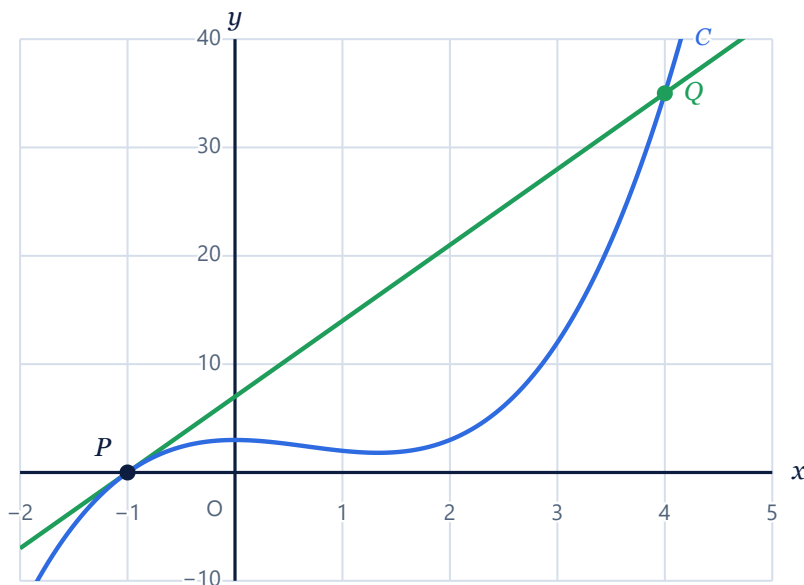
12

The curve C has equation $y = x^3 - 2x^2 + 3$. The point P on C has x -coordinate -1

a. Show that the tangent to C at P has equation $y = 7x + 7$

b. The tangent meets C again at the point Q . Show that the x -coordinates of P and Q satisfy $x^3 - 2x^2 - 7x - 4 = 0$ and hence find the coordinates of Q

c. Find the exact length of PQ , giving your answer as a simplified surd.



(9 marks)

a)

$$y = (-1)^3 - 2(-1)^2 + 3 = 0$$

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{dy}{dx} = 3 + 4 = 7$$

$$y - 0 = 7(x + 1)$$

$$y = 7x + 7$$

$$\boxed{y = 7x + 7}$$

b)

Solve the curve and tangent together

$$x^3 - 2x^2 + 3 = 7x + 7$$

$$x^3 - 2x^2 - 7x - 4 = 0$$

The line touches at $x = -1$, so $(x + 1)^2$ is a factor

$$(x + 1)^2(x - 4) = 0$$

$$\text{Check: } (x^2 + 2x + 1)(x - 4) = x^3 - 2x^2 - 7x - 4$$

$$x = 4$$

$$y = 7(4) + 7 = 35$$

$$Q(4, 35)$$

c)

$$PQ = \sqrt{((4 - (-1))^2 + (35 - 0)^2)}$$

$$PQ = \sqrt{25 + 1225}$$

$$PQ = \sqrt{1250}$$

$$PQ = \sqrt{625} \times \sqrt{2}$$

$$PQ = 25\sqrt{2}$$

$$25\sqrt{2}$$