



1 The vectors **a** and **b** are given by $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

a. Find $2\mathbf{a} - 3\mathbf{b}$, giving your answer in terms of **i** and **j**

b. Find $|\mathbf{a}|$

(3 marks)

a)

Write in i and j form

$$\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$$

$$2\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$$

$$3\mathbf{b} = -6\mathbf{i} + 15\mathbf{j}$$

$$2\mathbf{a} - 3\mathbf{b} = (6 - (-6))\mathbf{i} + (-8 - 15)\mathbf{j}$$

$$2\mathbf{a} - 3\mathbf{b} = 12\mathbf{i} - 23\mathbf{j}$$

$$12\mathbf{i} - 23\mathbf{j}$$

b)

Use Pythagoras on the components

$$|\mathbf{a}| = \sqrt{3^2 + (-4)^2}$$

$$|\mathbf{a}| = \sqrt{9 + 16}$$

$$|\mathbf{a}| = \sqrt{25}$$

$$|\mathbf{a}| = 5$$

$$|\mathbf{a}| = 5$$

2

The vector $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$

- Find a unit vector in the direction of $\mathbf{v}$
- Find the angle between $\mathbf{v}$ and the positive $\mathbf{i}$ direction.

(4 marks)

a)

Find the magnitude first

$$\begin{aligned} |\mathbf{v}| &= \sqrt{5^2 + (-12)^2} \\ |\mathbf{v}| &= \sqrt{169} \\ |\mathbf{v}| &= 13 \end{aligned}$$

Divide $\mathbf{v}$ by its magnitude

$$\text{unit vector} = \frac{1}{13}(5\mathbf{i} - 12\mathbf{j})$$

$$\boxed{\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}}$$

b)

Sketch: 5 across, 12 down, so $\mathbf{v}$ points below the $\mathbf{i}$ direction

$$\tan \theta = \frac{12}{5}$$

$$\theta = \tan^{-1}(2.4)$$

$$\theta = 67.38\dots^\circ$$

$$\boxed{67.4^\circ \text{ below the positive } \mathbf{i} \text{ direction}}$$

3

Relative to a fixed origin O , the point A has position vector $2\mathbf{i} - \mathbf{j}$ and the point B has position vector $-4\mathbf{i} + 7\mathbf{j}$

- Find $\vec{AB}$
- Find the distance AB
- Find the position vector of the midpoint of AB

(4 marks)

a)

$$\text{vector } AB = OB - OA$$

$$AB = (-4\mathbf{i} + 7\mathbf{j}) - (2\mathbf{i} - \mathbf{j})$$

$$AB = -6\mathbf{i} + 8\mathbf{j}$$

$$\boxed{-6\mathbf{i} + 8\mathbf{j}}$$

b)

$$|AB| = \sqrt{(-6)^2 + 8^2}$$

$$|AB| = \sqrt{36 + 64}$$

$$|AB| = \sqrt{100}$$

$$|AB| = 10$$

$$\boxed{10}$$

c)

$$\text{Midpoint} = \frac{1}{2}(OA + OB)$$

$$OM = \frac{1}{2}(2\mathbf{i} - \mathbf{j} - 4\mathbf{i} + 7\mathbf{j})$$

$$OM = \frac{1}{2}(-2\mathbf{i} + 6\mathbf{j})$$

$$OM = -\mathbf{i} + 3\mathbf{j}$$

$$\boxed{-\mathbf{i} + 3\mathbf{j}}$$

4

The vectors $\mathbf{p} = (k + 2)\mathbf{i} + 6\mathbf{j}$ and $\mathbf{q} = 2\mathbf{i} + (k + 3)\mathbf{j}$, where k is a constant.

Given that $\mathbf{p}$ is parallel to $\mathbf{q}$, find the possible values of k .

(4 marks)

Parallel means $\mathbf{p} = \lambda\mathbf{q}$, so the components are in the same ratio

$$\frac{(k + 2)}{2} = \frac{6}{(k + 3)}$$

Cross multiply

$$(k + 2)(k + 3) = 12$$

$$k^2 + 5k + 6 = 12$$

$$k^2 + 5k - 6 = 0$$

Factorise

$$(k + 6)(k - 1) = 0$$

$$k = 1 \text{ or } k = -6$$

Check

$$k = 1: \mathbf{p} = 3\mathbf{i} + 6\mathbf{j}, \mathbf{q} = 2\mathbf{i} + 4\mathbf{j}, \mathbf{p} = 1.5\mathbf{q}$$

$$k = -6: \mathbf{p} = -4\mathbf{i} + 6\mathbf{j}, \mathbf{q} = 2\mathbf{i} - 3\mathbf{j}, \mathbf{p} = -2\mathbf{q}$$

$k = 1 \text{ or } k = -6$

5

The vector $\mathbf{c} = 2t\mathbf{i} + (t - 5)\mathbf{j}$, where t is a constant.

Given that $|\mathbf{c}| = \sqrt{65}$

- find the possible values of t
- find, for the positive value of t , a unit vector in the direction of $\mathbf{c}$

(5 marks)

a)

Square the magnitude to remove the root

$$|\mathbf{c}|^2 = (2t)^2 + (t - 5)^2$$

$$65 = 4t^2 + t^2 - 10t + 25$$

$$0 = 5t^2 - 10t - 40$$

$$0 = t^2 - 2t - 8$$

$$0 = (t - 4)(t + 2)$$

$$t = 4 \text{ or } t = -2$$

b)

Substitute $t = 4$

$$\mathbf{c} = 8\mathbf{i} + (4 - 5)\mathbf{j}$$

$$\mathbf{c} = 8\mathbf{i} - \mathbf{j}$$

Divide by $|\mathbf{c}| = \sqrt{65}$

$$\frac{1}{\sqrt{65}}(8\mathbf{i} - \mathbf{j})$$

6

Relative to a fixed origin O , the points A and B have position vectors

$$\vec{OA} = 3\mathbf{i} + \mathbf{j} \quad \text{and} \quad \vec{OB} = -\mathbf{i} + 4\mathbf{j}$$

a. Find the size of angle AOB , giving your answer in degrees to one decimal place.

b. Find the exact area of triangle OAB

(6 marks)

a)

Find the three side lengths

$$|OA| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$|OB| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$AB = OB - OA = -4\mathbf{i} + 3\mathbf{j}$$

$$|AB| = \sqrt{16 + 9} = 5$$

Cosine rule for the angle at O

$$\cos AOB = \frac{(10 + 17 - 25)}{(2 \times \sqrt{10} \times \sqrt{17})}$$

$$\cos AOB = \frac{2}{(2\sqrt{170})}$$

$$\cos AOB = \frac{1}{\sqrt{170}}$$

$$AOB = 85.60\dots^\circ$$

$$\boxed{85.6^\circ}$$

b)

Use $\sin^2 \theta + \cos^2 \theta = 1$ to keep it exact

$$\sin AOB = \sqrt{1 - \frac{1}{170}}$$

$$\sin AOB = \sqrt{\frac{169}{170}}$$

$$\sin AOB = \frac{13}{\sqrt{170}}$$

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$\text{area} = \frac{1}{2} \times \sqrt{10} \times \sqrt{17} \times \frac{13}{\sqrt{170}}$$

$$\text{area} = \frac{1}{2} \times \sqrt{170} \times \frac{13}{\sqrt{170}}$$

$$\text{area} = \frac{13}{2}$$

7

Relative to a fixed origin O , the points A , B , C and D have position vectors

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{b} = 5\mathbf{i} + 4\mathbf{j}, \quad \mathbf{c} = 8\mathbf{i} + 10\mathbf{j}, \quad \mathbf{d} = 2\mathbf{i} + 7\mathbf{j}$$

a. Show that $ABCD$ is a trapezium, giving reasons for your answer.

b. Find the ratio $AB : DC$

(4 marks)

a)

Find vectors for opposite sides

$$\mathbf{AB} = (5\mathbf{i} + 4\mathbf{j}) - (\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{OC} = (8\mathbf{i} + 10\mathbf{j}) - (2\mathbf{i} + 7\mathbf{j}) = 6\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{OC} = \frac{3}{2}(4\mathbf{i} + 2\mathbf{j}) = \frac{3}{2}\mathbf{AB}$$

so AB is parallel to OC

$$\mathbf{AD} = (2\mathbf{i} + 7\mathbf{j}) - (\mathbf{i} + 2\mathbf{j}) = \mathbf{i} + 5\mathbf{j}$$

$$\mathbf{BC} = (8\mathbf{i} + 10\mathbf{j}) - (5\mathbf{i} + 4\mathbf{j}) = 3\mathbf{i} + 6\mathbf{j}$$

$1 : 5$ is not the same ratio as $3 : 6$, so AD is not parallel to BC

one pair of parallel sides only

$ABCD$ is a trapezium

b)

$$\mathbf{OC} = \frac{3}{2}\mathbf{AB}$$

$$\mathbf{AB} : \mathbf{OC} = 2 : 3$$

8

Relative to a fixed origin O , the point A has position vector $4\mathbf{i} - 7\mathbf{j}$ and the point B has position vector $14\mathbf{i} + 8\mathbf{j}$

The point P lies on the line segment AB such that $AP : PB = 3 : 2$

a. Find the position vector of P

b. Find $|\vec{OP}|$, giving your answer as a simplified surd.

(4 marks)

a)

AP is $\frac{3}{5}$ of AB

$$AB = (14\mathbf{i} + 8\mathbf{j}) - (4\mathbf{i} - 7\mathbf{j})$$

$$AB = 10\mathbf{i} + 15\mathbf{j}$$

$$AP = \frac{3}{5}(10\mathbf{i} + 15\mathbf{j})$$

$$AP = 6\mathbf{i} + 9\mathbf{j}$$

$$OP = OA + AP$$

$$OP = 4\mathbf{i} - 7\mathbf{j} + 6\mathbf{i} + 9\mathbf{j}$$

$$OP = 10\mathbf{i} + 2\mathbf{j}$$

$$\boxed{10\mathbf{i} + 2\mathbf{j}}$$

b)

$$|OP| = \sqrt{10^2 + 2^2}$$

$$|OP| = \sqrt{104}$$

$$|OP| = \sqrt{4} \times \sqrt{26}$$

$$|OP| = 2\sqrt{26}$$

$$\boxed{2\sqrt{26}}$$

9

In triangle OAB , $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$

The point M is the midpoint of OA . The point N lies on AB such that $AN : NB = 2 : 1$

a. Find $\vec{MN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, giving your answer in its simplest form.

The point P is such that $\vec{OP} = 2\mathbf{b}$

b. Show that M , N and P lie on a straight line.

c. Find the ratio $MN : NP$

(6 marks)

a)

Go from O to N along OA then part of AB

$$\vec{AB} = \mathbf{b} - \mathbf{a}$$

$$\vec{ON} = \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$$

$$\vec{ON} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$$

$$\vec{OM} = \frac{1}{2}\mathbf{a}$$

$$\vec{MN} = \vec{ON} - \vec{OM}$$

$$\vec{MN} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b} - \frac{1}{2}\mathbf{a}$$

$$\vec{MN} = -\frac{1}{6}\mathbf{a} + \frac{2}{3}\mathbf{b}$$

$$\boxed{\vec{MN} = -\frac{1}{6}\mathbf{a} + \frac{2}{3}\mathbf{b}}$$

b)

$$\vec{MP} = \vec{OP} - \vec{OM}$$

$$\vec{MP} = 2\mathbf{b} - \frac{1}{2}\mathbf{a}$$

$$\vec{MP} = 3\left(-\frac{1}{6}\mathbf{a} + \frac{2}{3}\mathbf{b}\right)$$

$$\vec{MP} = 3\vec{MN}$$

$\vec{MP}$ is parallel to $\vec{MN}$ and both pass through M

$$\boxed{M, N \text{ and } P \text{ lie on a straight line}}$$

c)

$$\vec{MP} = 3\vec{MN}$$

$$\vec{NP} = \vec{MP} - \vec{MN} = 2\vec{MN}$$

$$MN : NP = 1 : 2$$

10

A park is modelled as a quadrilateral $OPQR$. Relative to the point O , the corners have position vectors

$$\vec{OP} = 6\mathbf{i}, \quad \vec{OQ} = 10\mathbf{i} + 9\mathbf{j}, \quad \vec{OR} = 8\mathbf{i} + 18\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north and each unit represents 10 metres.

- Show that the side PQ is parallel to the side OR .
- A straight path runs from O to Q . Find the length of the path in metres.
- Find the size of angle OPQ .
- Give one limitation of the model.

(8 marks)

a)

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$\vec{PQ} = 10\mathbf{i} + 9\mathbf{j} - 6\mathbf{i}$$

$$\vec{PQ} = 4\mathbf{i} + 9\mathbf{j}$$

$$\vec{OR} = 8\mathbf{i} + 18\mathbf{j} = 2(4\mathbf{i} + 9\mathbf{j})$$

$$\vec{OR} = 2\vec{PQ}$$

PQ is parallel to OR

b)

$$|\vec{OQ}| = \sqrt{10^2 + 9^2}$$

$$|\vec{OQ}| = \sqrt{181}$$

$$|\vec{OQ}| = 13.453... \text{ units}$$

Each unit is 10 m

$$\text{Length} = 134.53... \text{ m}$$

135 m

c)

Side lengths of triangle OPQ

$$|\vec{OP}| = 6$$

$$|\vec{PQ}| = \sqrt{4^2 + 9^2} = \sqrt{97}$$

$$|\vec{OQ}| = \sqrt{181}$$

Cosine rule for the angle at P

$$\cos OPQ = \frac{(36 + 97 - 181)}{(2 \times 6 \times \sqrt{97})}$$

$$\cos OPQ = \frac{-48}{(12\sqrt{97})}$$

$$\cos OPQ = -0.4061...$$

$$\text{OPQ} = 113.96...^\circ$$

$$114.0^\circ$$

d)

the edges will not be perfectly straight and the ground is not flat

11

Relative to a fixed origin O , the points A , B and C have position vectors

$$\mathbf{i} + 2\mathbf{j}, \quad 4\mathbf{i} + 6\mathbf{j} \quad \text{and} \quad p\mathbf{i} - 3\mathbf{j}$$

respectively, where p is a constant. Given that angle $ABC = 90^\circ$

- use Pythagoras' theorem to find the value of p
- find the area of triangle ABC
- The point D is such that $ABCD$ is a rectangle. Find the position vector of D

(7 marks)

a)

$$\mathbf{AB} = (4\mathbf{i} + 6\mathbf{j}) - (\mathbf{i} + 2\mathbf{j}) = 3\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{BC} = (p\mathbf{i} - 3\mathbf{j}) - (4\mathbf{i} + 6\mathbf{j}) = (p - 4)\mathbf{i} - 9\mathbf{j}$$

$$\mathbf{AC} = (p\mathbf{i} - 3\mathbf{j}) - (\mathbf{i} + 2\mathbf{j}) = (p - 1)\mathbf{i} - 5\mathbf{j}$$

Right angle at B , so AC is the hypotenuse

$$|\mathbf{AC}|^2 = |\mathbf{AB}|^2 + |\mathbf{BC}|^2$$

$$(p - 1)^2 + 25 = 25 + (p - 4)^2 + 81$$

$$p^2 - 2p + 1 = p^2 - 8p + 16 + 81$$

$$6p = 96$$

$$p = 16$$

$$\boxed{p = 16}$$

b)

$$|\mathbf{AB}| = \sqrt{9 + 16} = 5$$

$$\mathbf{BC} = 12\mathbf{i} - 9\mathbf{j}$$

$$|\mathbf{BC}| = \sqrt{144 + 81} = 15$$

$$\text{area} = \frac{1}{2} \times 5 \times 15$$

$$\text{area} = 37.5$$

$$\boxed{37.5}$$

c)

In a rectangle $\mathbf{AD} = \mathbf{BC}$

$$\mathbf{OD} = \mathbf{OA} + \mathbf{BC}$$

$$\mathbf{OD} = \mathbf{i} + 2\mathbf{j} + 12\mathbf{i} - 9\mathbf{j}$$

$$\mathbf{OD} = 13\mathbf{i} - 7\mathbf{j}$$

$$\boxed{13\mathbf{i} - 7\mathbf{j}}$$

In triangle OAB , $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$

The point P lies on OA such that $OP : PA = 1 : 2$ and the point Q is the midpoint of AB . The lines OQ and BP meet at the point X .

- Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, (i) $\vec{OQ}$ (ii) $\vec{BP}$
- Using a vector method, find the ratio $OX : XQ$
- Find the ratio $BX : XP$

(8 marks)

a)

$$\vec{OQ} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

$$\vec{OQ} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

$$\vec{BP} = \vec{OP} - \vec{OB}$$

$$\vec{BP} = \frac{1}{3}\mathbf{a} - \mathbf{b}$$

$$\vec{OQ} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}, \vec{BP} = \frac{1}{3}\mathbf{a} - \mathbf{b}$$

b)

Write OX two ways

$$\vec{OX} = \lambda \vec{OQ} = \frac{\lambda}{2}\mathbf{a} + \frac{\lambda}{2}\mathbf{b}$$

$$\vec{OX} = \vec{OB} + \mu \vec{BP} = \mathbf{b} + \mu\left(\frac{1}{3}\mathbf{a} - \mathbf{b}\right)$$

$$\vec{OX} = \frac{\mu}{3}\mathbf{a} + (1 - \mu)\mathbf{b}$$

$\mathbf{a}$ and $\mathbf{b}$ are not parallel, so compare coefficients

$$\frac{\lambda}{2} = \frac{\mu}{3}$$

$$\frac{\lambda}{2} = 1 - \mu$$

$$\frac{\mu}{3} = 1 - \mu$$

$$\mu = 3 - 3\mu$$

$$\mu = \frac{3}{4}$$

$$\lambda = \frac{2\mu}{3} = \frac{1}{2}$$

$$OX = \frac{1}{2} OQ$$

$$OX : XQ = 1 : 1$$

c)

$$BX = \mu BP = \frac{3}{4} BP$$

$$BX : XP = 3 : 1$$