



1 Simplify fully

a. $\frac{3x^2 - 12}{x^2 + x - 6}$

b. $\frac{x^2 + 5x}{x^2 - 1} \div \frac{x^2 + 3x - 10}{x^2 - x}$

(5 marks)

a) Factorise top and bottom

$$\begin{aligned}\frac{3x^2 - 12}{x^2 + x - 6} &= \frac{3(x^2 - 4)}{(x + 3)(x - 2)} \\ &= \frac{3(x - 2)(x + 2)}{(x + 3)(x - 2)}\end{aligned}$$

Cancel $(x - 2)$

$$= \frac{3(x + 2)}{x + 3}$$

$$\boxed{\frac{3(x + 2)}{x + 3}}$$

b) Dividing: flip the second fraction and multiply

$$\frac{x^2 + 5x}{x^2 - 1} \times \frac{x^2 - x}{x^2 + 3x - 10}$$

Factorise everything

$$= \frac{x(x + 5)}{(x - 1)(x + 1)} \times \frac{x(x - 1)}{(x + 5)(x - 2)}$$

Cancel $(x + 5)$ and $(x - 1)$

$$= \frac{x^2}{(x + 1)(x - 2)}$$

$$\boxed{\frac{x^2}{(x + 1)(x - 2)}}$$

2

Express $\frac{2x^2 + 6x - 2}{x(x-1)(x+2)}$ in partial fractions.

(4 marks)

Write as $\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$

$$2x^2 + 6x - 2 \equiv A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Put $x = 0$

$$-2 = A(-1)(2)$$

$$A = 1$$

Put $x = 1$

$$6 = B(1)(3)$$

$$B = 2$$

Put $x = -2$

$$8 - 12 - 2 = C(-2)(-3)$$

$$-6 = 6C$$

$$C = -1$$

$$\boxed{\frac{1}{x} + \frac{2}{x-1} - \frac{1}{x+2}}$$

3

Given that

$$2x^3 + \frac{3x^2 - 5x + 4}{x+2} \equiv Ax^2 + Bx + C + \frac{D}{x+2}, \quad x \neq -2$$

find the values of the constants A , B , C and D .

(4 marks)

Divide $2x^3 + 3x^2 - 5x + 4$ by $(x+2)$, one term at a time

$$2x^3 \div x = 2x^2$$

$$2x^3 + 3x^2 - 5x + 4 - 2x^2(x+2) = -x^2 - 5x + 4$$

$$-x^2 \div x = -x$$

$$-x^2 - 5x + 4 - (-x)(x+2) = -3x + 4$$

$$-3x \div x = -3$$

$$-3x + 4 - (-3)(x+2) = 10$$

Quotient $2x^2 - x - 3$, remainder 10

$$\frac{2x^3 + 3x^2 - 5x + 4}{x+2} = 2x^2 - x - 3 + \frac{10}{x+2}$$

$$\boxed{A = 2, B = -1, C = -3, D = 10}$$

4

$$f(x) = 2 - \frac{4}{x+1} - \frac{16}{x^2 - 2x - 3}, \quad x > 3$$

a. Show that $f(x) = \frac{2x-10}{x-3}$

b. Hence solve the equation $f(x) = x - 6$

(7 marks)

a) Factorise $x^2 - 2x - 3 = (x+1)(x-3)$

Common denominator $(x+1)(x-3)$

$$f(x) = \frac{2(x+1)(x-3) - 4(x-3) - 16}{(x+1)(x-3)}$$

Expand the numerator

$$= \frac{2x^2 - 4x - 6 - 4x + 12 - 16}{(x+1)(x-3)}$$

$$= \frac{2x^2 - 8x - 10}{(x+1)(x-3)}$$

$$= \frac{2(x-5)(x+1)}{(x+1)(x-3)}$$

Cancel $(x+1)$

$$= \frac{2(x-5)}{x-3}$$

$$f(x) = \frac{2x-10}{x-3}$$

b) Multiply both sides by $(x-3)$

$$2x - 10 = (x-6)(x-3)$$

$$2x - 10 = x^2 - 9x + 18$$

$$0 = x^2 - 11x + 28$$

$$0 = (x-4)(x-7)$$

Both are in the domain $x > 3$

$$x = 4 \text{ or } x = 7$$

5

Express $\frac{3x^2 - x + 8}{(x-3)(x+1)^2}$ in partial fractions.

(4 marks)

Repeated factor: write as $\frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$$3x^2 - x + 8 \equiv A(x+1)^2 + B(x-3)(x+1) + C(x-3)$$

Put $x = 3$

$$27 - 3 + 8 = 16A$$

$$A = 2$$

Put $x = -1$

$$3 + 1 + 8 = -4C$$

$$C = -3$$

Compare coefficients of x^2

$$3 = A + B$$

$$B = 1$$

$$\boxed{\frac{2}{x-3} + \frac{1}{x+1} - \frac{3}{(x+1)^2}}$$

6

Express $\frac{2x^2 + 1}{(x-2)(x+1)}$ in the form $A + \frac{B}{x-2} + \frac{C}{x+1}$, where A , B and C are constants to be found.

(4 marks)

Improper: top and bottom are both degree 2, so divide first

$$(x-2)(x+1) = x^2 - x - 2$$

$$2x^2 + 1 = 2(x^2 - x - 2) + 2x + 5$$

$$\frac{2x^2 + 1}{(x-2)(x+1)} = 2 + \frac{2x + 5}{(x-2)(x+1)}$$

Partial fractions on the remainder

$$2x + 5 \equiv B(x+1) + C(x-2)$$

Put $x = 2$

$$9 = 3B$$

$$B = 3$$

Put $x = -1$

$$3 = -3C$$

$$C = -1$$

$$\boxed{2 + \frac{3}{x-2} - \frac{1}{x+1}}$$

7

Given that

$$x^4 + \frac{2x^3 - 3x^2 + x - 5}{x^2 + x - 1} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 + x - 1}$$

find the values of the constants a, b, c, d and e .

(4 marks)

Divide by $x^2 + x - 1$, one term at a time

$$x^4 \div x^2 = x^2$$

$$x^4 + 2x^3 - 3x^2 + x - 5 - x^2(x^2 + x - 1) = x^3 - 2x^2 + x - 5$$

$$x^3 \div x^2 = x$$

$$x^3 - 2x^2 + x - 5 - x(x^2 + x - 1) = -3x^2 + 2x - 5$$

$$-3x^2 \div x^2 = -3$$

$$-3x^2 + 2x - 5 - (-3)(x^2 + x - 1) = 5x - 8$$

*Remainder is linear, so stop*Quotient $x^2 + x - 3$, remainder $5x - 8$

$a = 1, b = 1, c = -3, d = 5, e = -8$

a. Express $\frac{4x+3}{(2x-1)(x+2)}$ in partial fractions.

b. Hence find the exact value of $\int_1^3 \frac{4x+3}{(2x-1)(x+2)} dx$, giving your answer in the form $\ln k$, where k is a rational number.

(6 marks)

a) Write as $\frac{A}{2x-1} + \frac{B}{x+2}$

$$4x+3 \equiv A(x+2) + B(2x-1)$$

Put $x = \frac{1}{2}$

$$5 = \frac{5}{2}A$$

$$A = 2$$

Put $x = -2$

$$-5 = -5B$$

$$B = 1$$

$$\frac{2}{2x-1} + \frac{1}{x+2}$$

b) Integrate each fraction

$$\int \left(\frac{2}{2x-1} + \frac{1}{x+2} \right) dx = \ln(2x-1) + \ln(x+2)$$

Limits from 1 to 3

$$= (\ln 5 + \ln 5) - (\ln 1 + \ln 3)$$

$$= 2 \ln 5 - \ln 3$$

Use Log Laws

$$= \ln \frac{25}{3}$$

$$\ln \frac{25}{3}$$

9

$$f(x) = \frac{kx + 6}{(x-1)(x+2)}, \quad x > 1, \text{ where } k \text{ is a positive constant.}$$

a. Express $f(x)$ in partial fractions in terms of k .

b. Given that $\int_2^{10} f(x) dx = 10 \ln 3$, find the value of k .

(7 marks)

a) Write as $\frac{A}{x-1} + \frac{B}{x+2}$

$$kx + 6 \equiv A(x+2) + B(x-1)$$

Put $x = 1$

$$k + 6 = 3A$$

$$A = \frac{k+6}{3}$$

Put $x = -2$

$$-2k + 6 = -3B$$

$$B = \frac{2k-6}{3}$$

$$f(x) = \frac{k+6}{3(x-1)} + \frac{2k-6}{3(x+2)}$$

b) Integrate

$$\int f(x) dx = A \ln(x-1) + B \ln(x+2)$$

Limits from 2 to 10

$$= A(\ln 9 - \ln 1) + B(\ln 12 - \ln 4)$$

$$= A \ln 9 + B \ln 3$$

$\ln 9 = 2 \ln 3$

$$= (2A + B) \ln 3$$

Set equal to $10 \ln 3$

$$2A + B = 10$$

$$\frac{2(k+6) + (2k-6)}{3} = 10$$

$$4k + 6 = 30$$

$$k = 6$$

$$k = 6$$

10

The function f is defined by

$$f(x) = \frac{3x^2 + 13x - 8}{(x-1)(x+3)}, \quad x > 1$$

a. Express $f(x)$ in the form $A + \frac{B}{x-1} + \frac{C}{x+3}$, where A , B and C are constants.b. Hence find $f'(x)$ and explain why f is a decreasing function.c. State the range of f .

(9 marks)

a) Improper, so divide first

$$(x-1)(x+3) = x^2 + 2x - 3$$

$$3x^2 + 13x - 8 = 3(x^2 + 2x - 3) + 7x + 1$$

$$7x + 1 \equiv B(x+3) + C(x-1)$$

Put $x = 1$

$$8 = 4B$$

$$B = 2$$

Put $x = -3$

$$-20 = -4C$$

$$C = 5$$

$$f(x) = 3 + \frac{2}{x-1} + \frac{5}{x+3}$$

b) Write as powers and differentiate

$$f(x) = 3 + 2(x-1)^{-1} + 5(x+3)^{-1}$$

$$\begin{aligned} f'(x) &= -2(x-1)^{-2} - 5(x+3)^{-2} \\ &= -\frac{2}{(x-1)^2} - \frac{5}{(x+3)^2} \end{aligned}$$

 $(x-1)^2 > 0$ and $(x+3)^2 > 0$, so both terms are negative $f'(x) < 0$ for all $x > 1$, so f is decreasing

$$f'(x) = -\frac{2}{(x-1)^2} - \frac{5}{(x+3)^2} < 0$$

c) As $x \rightarrow 1$ from above, $\frac{2}{x-1} \rightarrow \infty$ As $x \rightarrow \infty$, both fractions are positive and $\rightarrow 0$ $f(x) \rightarrow 3$ from above, never reaching 3

$$f(x) > 3$$

11

$$f(x) = \frac{18x^2 - 9x + 8}{(3x - 1)^2(x + 2)}, \quad x > \frac{1}{3}$$

Given that $f(x)$ can be written in the form

$$\frac{A}{3x - 1} + \frac{B}{(3x - 1)^2} + \frac{C}{x + 2}$$

where A , B and C are constants,

a. (i) find the value of B and the value of C ,

(ii) show that $A = 0$

b. Hence find the exact value of $\int_1^2 f(x) \, dx$, giving your answer in the form $p + q \ln r$, where p , q and r are rational.

(8 marks)

a) Multiply through by $(3x - 1)^2(x + 2)$

$$18x^2 - 9x + 8 \equiv A(3x - 1)(x + 2) + B(x + 2) + C(3x - 1)^2$$

(i) Put $x = \frac{1}{3}$

$$2 - 3 + 8 = \frac{7}{3}B$$

$$B = 3$$

Put $x = -2$

$$72 + 18 + 8 = 49C$$

$$C = 2$$

$$B = 3, C = 2$$

(ii) Compare coefficients of x^2

$$18 = 3A + 9C$$

$$18 = 3A + 18$$

$$A = 0$$

$$A = 0 \text{ (shown)}$$

b) So $f(x) = 3(3x - 1)^{-2} + \frac{2}{x + 2}$

Reverse chain rule: divide by the 3 from the bracket

$$\begin{aligned} \int 3(3x - 1)^{-2} \, dx &= \frac{3(3x - 1)^{-1}}{(-1) \times 3} \\ &= -\frac{1}{3x - 1} \end{aligned}$$

$$\int \frac{2}{x + 2} \, dx = 2 \ln(x + 2)$$

Limits from 1 to 2

$$\begin{aligned} \left[-\frac{1}{3x-1} + 2 \ln(x+2) \right] &= \left(-\frac{1}{5} + 2 \ln 4 \right) - \left(-\frac{1}{2} + 2 \ln 3 \right) \\ &= \frac{3}{10} + 2 \ln \frac{4}{3} \end{aligned}$$

$$\boxed{\frac{3}{10} + 2 \ln \frac{4}{3}}$$

12

Rainwater flows into a garden water butt. The rate of flow, R litres per hour, t hours after the rain starts is modelled by

$$R = \frac{30t}{(t+1)(t+6)}, \quad 0 \leq t \leq 10$$

- Express $\frac{30t}{(t+1)(t+6)}$ in partial fractions.
- Hence find, according to the model, the exact volume of water that flows into the water butt in the first 4 hours. Give your answer in the form $a \ln 5 - b \ln 3$, where a and b are integers.
- Using your answer to part (a), find the greatest rate of flow according to the model, giving your answer to 3 significant figures.

(10 marks)

a) Write as $\frac{A}{t+1} + \frac{B}{t+6}$

$$30t \equiv A(t+6) + B(t+1)$$

Put $t = -1$

$$-30 = 5A$$

$$A = -6$$

Put $t = -6$

$$-180 = -5B$$

$$B = 36$$

$$R = \frac{36}{t+6} - \frac{6}{t+1}$$

b) Volume = $\int R \, dt$, Limits from 0 to 4

$$\int \left(\frac{36}{t+6} - \frac{6}{t+1} \right) dt = 36 \ln(t+6) - 6 \ln(t+1)$$

$$V = (36 \ln 10 - 6 \ln 5) - (36 \ln 6 - 6 \ln 1)$$

$$= 36 \ln \frac{10}{6} - 6 \ln 5$$

$$= 36 \ln 5 - 36 \ln 3 - 6 \ln 5$$

$$= 30 \ln 5 - 36 \ln 3$$

$$\approx 8.73 \text{ litres}$$

$$30 \ln 5 - 36 \ln 3 \text{ litres}$$

c) Differentiate the partial fractions

$$\frac{dR}{dt} = -\frac{36}{(t+6)^2} + \frac{6}{(t+1)^2}$$

Greatest rate when $\frac{dR}{dt} = 0$

$$6(t+6)^2 = 36(t+1)^2$$

$$(t+6)^2 = 6(t+1)^2$$

$t > 0$, so take the positive root

$$t+6 = \sqrt{6}(t+1)$$

$$t(\sqrt{6}-1) = 6 - \sqrt{6}$$

$$t = \frac{\sqrt{6}(\sqrt{6}-1)}{\sqrt{6}-1} = \sqrt{6}$$

$$R = \frac{36}{\sqrt{6+6}} - \frac{6}{\sqrt{6+1}}$$

$$= 2.521\dots$$

$R(0) = 0$ and $R(10) = 1.70$, so this is the greatest rate

2.52 litres per hour