



1

- a. Find the first four terms, in ascending powers of x , of the binomial expansion of $(1 + 3x)^{-2}$, giving each term in its simplest form.
- b. State the range of values of x for which the expansion is valid.

(4 marks)

a.)

$$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \frac{n(n-1)(n-2)}{3!}y^3 + \dots$$

$$n = -2, y = 3x$$

$$(1 + 3x)^{-2} = 1 + (-2)(3x) + \frac{(-2)(-3)}{2}(3x)^2 + \frac{(-2)(-3)(-4)}{6}$$

$$(3x)^3$$

$$= 1 - 6x + 3(9x^2) - 4(27x^3)$$

$$\boxed{1 - 6x + 27x^2 - 108x^3}$$

b.)

$$\text{Valid when } |3x| < 1$$

$$\boxed{|x| < \frac{1}{3}}$$

2

In the binomial expansion of $(1 + kx)^{-3}$, where k is a negative constant, the coefficient of x^2 is 54

a. Find the value of k .

b. Find the coefficient of x^3 in the expansion.

(4 marks)

a)

x^2 term

$$\frac{(-3)(-4)}{2!}(kx)^2 = 6k^2x^2$$

$$6k^2 = 54$$

$$k^2 = 9$$

k is negative

$$k = -3$$

b)

x^3 term

$$\frac{(-3)(-4)(-5)}{3!}(kx)^3 = -10k^3x^3$$

$$-10(-3)^3 = -10 \times (-27)$$

$$270$$

3

- a. Find the first four terms, in ascending powers of x , of the binomial expansion of $\frac{1}{\sqrt{4+x}}$, giving each coefficient as a simplified fraction.
- b. State the range of values of x for which the expansion is valid.

(5 marks)

a)

Take out the factor $4^{-\frac{1}{2}} = \frac{1}{2}$

$$(4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}} \left(1 + \frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2} \left(1 + \frac{x}{4}\right)^{-\frac{1}{2}}$$

Expand with $n = -\frac{1}{2}$, $y = \frac{x}{4}$

$$(1+y)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)y + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}y^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}y^3$$

$$= 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3$$

$$= 1 - \frac{x}{8} + \frac{3x^2}{128} - \frac{5x^3}{1024}$$

Multiply by $\frac{1}{2}$

$\frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256} - \frac{5x^3}{1024}$

b)

Valid when $\left|\frac{x}{4}\right| < 1$

$x < 4$

4

- a. Find the first four terms, in ascending powers of x , of the binomial expansion of $\sqrt{1+8x}$, giving each term in its simplest form.
- b. Explain why substituting $x = \frac{1}{8}$ into this expansion would not give a valid approximation for $\sqrt{2}$
- c. By substituting $x = \frac{1}{32}$ into your expansion, find an approximation for $\sqrt{5}$, giving your answer to 4 decimal places.

(7 marks)

a)

$$n = \frac{1}{2}, y = 8x$$

$$\begin{aligned}(1+8x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(8x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(8x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(8x)^3 \\ &= 1 + 4x - \frac{1}{8}(64x^2) + \frac{1}{16}(512x^3)\end{aligned}$$

$$1 + 4x - 8x^2 + 32x^3$$

b)

Valid only when $|8x| < 1$

$$|x| < \frac{1}{8}$$

$x = \frac{1}{8}$ is not inside $|x| < \frac{1}{8}$, so the expansion is not valid there

c)

$$\begin{aligned}\sqrt{1+8 \times \frac{1}{32}} &= \sqrt{\left(\frac{5}{4}\right)} \\ &= \frac{\sqrt{5}}{2}\end{aligned}$$

$$\begin{aligned}\frac{\sqrt{5}}{2} &\approx 1 + \frac{4}{32} - \frac{8}{1024} + \frac{32}{32768} \\ &= 1.1181640...\end{aligned}$$

$$\begin{aligned}\sqrt{5} &\approx 2 \times 1.1181640... \\ &= 2.236328...\end{aligned}$$

$$\sqrt{5} \approx 2.2363$$

5

- a. Show that the binomial expansion of $(8 - 3x)^{\frac{1}{3}}$, in ascending powers of x up to and including the term in x^2 , is

$$2 - \frac{1}{4}x - \frac{1}{32}x^2$$

- b. State the range of values of x for which the expansion is valid.

- c. Use the expansion with a suitable value of x to find an estimate for $\sqrt[3]{7.97}$, giving your answer to 6 decimal places.

(6 marks)

a)

Take out $8^{\frac{1}{3}} = 2$

$$(8 - 3x)^{\frac{1}{3}} = 2\left(1 - \frac{3x}{8}\right)^{\frac{1}{3}}$$

$$n = \frac{1}{3}, y = -\frac{3x}{8}$$

$$(1 + y)^{\frac{1}{3}} = 1 + \frac{1}{3}y + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}y^2$$

$$= 1 + \frac{1}{3}y - \frac{1}{9}y^2$$

$$= 1 + \frac{1}{3}\left(-\frac{3x}{8}\right) - \frac{1}{9}\left(\frac{9x^2}{64}\right)$$

$$= 1 - \frac{x}{8} - \frac{x^2}{64}$$

$$2\left(1 - \frac{x}{8} - \frac{x^2}{64}\right) = 2 - \frac{1}{4}x - \frac{1}{32}x^2$$

shown

b)

Valid when $\left|\frac{3x}{8}\right| < 1$

$$|x| < \frac{8}{3}$$

c)

Choose x so that $8 - 3x = 7.97$

$$3x = 0.03$$

$$x = 0.01$$

$$\sqrt[3]{7.97} \approx 2 - \frac{0.01}{4} - \frac{0.01^2}{32}$$

$$= 2 - 0.0025 - 0.000003125$$

$$= 1.997496875$$

$$\sqrt[3]{7.97} \approx 1.997497$$

6

The binomial expansion of $(a + bx)^{-2}$, where a and b are positive constants, in ascending powers of x , begins

$$\frac{1}{9} - \frac{4}{27}x + cx^2 + \dots$$

- Find the values of a and b .
- Find the value of c .
- State the range of values of x for which the expansion is valid.

(6 marks)

a)

$$\begin{aligned}(a + bx)^{-2} &= a^{-2} \left(1 + \frac{b}{a}x\right)^{-2} \\&= \frac{1}{a^2} \left(1 - 2\left(\frac{b}{a}x\right) + \frac{(-2)(-3)}{2} \left(\frac{b}{a}x\right)^2 + \dots\right) \\&= \frac{1}{a^2} - \frac{2b}{a^3}x + \frac{3b^2}{a^4}x^2 + \dots\end{aligned}$$

Compare constant terms

$$\frac{1}{a^2} = \frac{1}{9}$$

$$a = 3$$

Compare x terms

$$\frac{2b}{27} = \frac{4}{27}$$

$$a = 3, b = 2$$

b)

$$\begin{aligned}c &= \frac{3b^2}{a^4} \\&= \frac{3 \times 4}{81}\end{aligned}$$

$$c = \frac{4}{27}$$

c)

Valid when $\left|\frac{2x}{3}\right| < 1$

$$|x| < \frac{3}{2}$$

7

- a. Find the binomial expansion of $(2 + x)^{-2}$, in ascending powers of x up to and including the term in x^2 , giving each coefficient as a simplified fraction.
- b. Hence find the first three terms, in ascending powers of x , of the expansion of $\frac{3 - x}{(2 + x)^2}$
- c. State the range of values of x for which this expansion is valid.

(6 marks)

a)

$$\begin{aligned}
 (2 + x)^{-2} &= 2^{-2} \left(1 + \frac{x}{2}\right)^{-2} \\
 &= \frac{1}{4} \left(1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2} \left(\frac{x}{2}\right)^2 + \dots\right) \\
 &= \frac{1}{4} \left(1 - x + \frac{3}{4}x^2 + \dots\right)
 \end{aligned}$$

$$\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2$$

b)

$$(3 - x) \left(\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 \right)$$

Constant term

$$3 \times \frac{1}{4} = \frac{3}{4}$$

 x terms

$$-\frac{3}{4}x - \frac{1}{4}x = -x$$

 x^2 terms

$$\frac{9}{16}x^2 + \frac{1}{4}x^2 = \frac{13}{16}x^2$$

$$\frac{3}{4} - x + \frac{13}{16}x^2$$

c)

Valid when $\left|\frac{x}{2}\right| < 1$

$$|x| < 2$$

a. Use binomial expansions to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2$$

b. By substituting $x = \frac{1}{9}$ into both sides, find an approximation for $\sqrt{5}$, giving your answer as a fraction in its simplest form.

c. Find the percentage error in this approximation, giving your answer to 2 significant figures.

(8 marks)

a)

Write the root as a product of two powers

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-x)^2$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

Multiply, keeping terms up to x^2

$$\begin{aligned} \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right) &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 \\ &\quad - \frac{1}{8}x^2 \\ &= 1 + x + \frac{1}{2}x^2 \end{aligned}$$

shown

b)

$$\frac{1 + \frac{1}{9}}{1 - \frac{1}{9}} = \frac{10}{8}$$

$$= \frac{5}{4}$$

$$\sqrt{\left(\frac{5}{4}\right)} = \frac{\sqrt{5}}{2}$$

$$1 + \frac{1}{9} + \frac{1}{2} \left(\frac{1}{81} \right) = \frac{162 + 18 + 1}{162}$$

$$= \frac{181}{162}$$

$$\frac{\sqrt{5}}{2} \approx \frac{181}{162}$$

$$\boxed{\sqrt{5} \approx \frac{181}{81}}$$

c)

$$\frac{181}{81} = 2.234567...$$

$$\sqrt{5} = 2.236067...$$

$$\% \text{ error} = \frac{2.236067... - 2.234567...}{2.236067...} \times 100$$

$$= 0.06708...$$

$$\boxed{0.067\%}$$

9

The binomial expansion of $(1 + ax)^n$, in ascending powers of x , begins

$$1 - 6x + 24x^2 + \dots$$

where a and n are constants and n is not a positive integer.

- Find the value of n and the value of a .
- Find the coefficient of x^3
- State the range of values of x for which the expansion is valid.

(6 marks)

a)

Compare coefficients

$$na = -6$$

$$\frac{n(n-1)}{2}a^2 = 24$$

$$n(n-1)a^2 = 48$$

Substitute $a = -\frac{6}{n}$

$$n(n-1) \times \frac{36}{n^2} = 48$$

$$\frac{36(n-1)}{n} = 48$$

$$36n - 36 = 48n$$

$$-12n = 36$$

$$n = -3$$

$$a = \frac{-6}{-3}$$

$$\boxed{n = -3, a = 2}$$

b)

$$\frac{(-3)(-4)(-5)}{3!} \times 2^3 = -10 \times 8$$

$$\boxed{-80}$$

c)

Valid when $|2x| < 1$

$$\boxed{|x| < \frac{1}{2}}$$

$$f(x) = \frac{4 + 5x}{(1 + 2x)(1 - x)}$$

- Express $f(x)$ in partial fractions.
- Hence find the series expansion of $f(x)$, in ascending powers of x up to and including the term in x^3 , giving each coefficient in its simplest form.
- State the range of values of x for which the expansion is valid, giving a reason.

(9 marks)

a)

$$\frac{4 + 5x}{(1 + 2x)(1 - x)} = \frac{A}{1 + 2x} + \frac{B}{1 - x}$$

$$4 + 5x = A(1 - x) + B(1 + 2x)$$

$$x = 1$$

$$9 = 3B$$

$$B = 3$$

$$x = -\frac{1}{2}$$

$$4 - \frac{5}{2} = \frac{3}{2}A$$

$$A = 1$$

$$f(x) = \frac{1}{1 + 2x} + \frac{3}{1 - x}$$

b)

$$(1 + 2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{6}(2x)^3$$

$$= 1 - 2x + 4x^2 - 8x^3$$

$$3(1 - x)^{-1} = 3(1 + x + x^2 + x^3)$$

$$= 3 + 3x + 3x^2 + 3x^3$$

Add

$$f(x) = 4 + x + 7x^2 - 5x^3 + \dots$$

$$4 + x + 7x^2 - 5x^3$$

c)

$$(1 + 2x)^{-1} \text{ needs } |x| < \frac{1}{2}; (1 - x)^{-1} \text{ needs } |x| < 1$$

Both must be valid, so take the narrower range

$$|x| < \frac{1}{2}$$

a. Find the binomial expansion of $\frac{1}{\sqrt{1+x^2}}$, in ascending powers of x up to and including the term in x^4

b. Hence find an approximation for

$$\int_0^{0.2} \frac{1}{\sqrt{1+x^2}} dx$$

giving your answer to 6 significant figures.

c. Explain why this approximation is likely to be very accurate.

(7 marks)

a)

$$n = -\frac{1}{2}, y = x^2$$

$$(1+x^2)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(x^2)^2$$

$$\boxed{1 - \frac{1}{2}x^2 + \frac{3}{8}x^4}$$

b)

$$\int \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4\right) dx = x - \frac{x^3}{6} + \frac{3x^5}{40}$$

Limits: from 0 to 0.2

$$\begin{aligned} \left[x - \frac{x^3}{6} + \frac{3x^5}{40}\right] &= 0.2 - \frac{0.008}{6} + \frac{3 \times 0.00032}{40} - 0 \\ &= 0.2 - 0.0013333... + 0.000024 \\ &= 0.1986906... \end{aligned}$$

$$\boxed{0.198691}$$

c)

$x \leq 0.2$ so $x^2 \leq 0.04$: well inside $|x| < 1$

the terms left out (x^6 and beyond) are tiny

In physics, the total energy E of a particle of mass m moving at speed v is

$$E = mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

where c is the speed of light and $0 \leq v < c$. The kinetic energy is $K = E - mc^2$

a. Use a binomial expansion to show that, when v is small compared with c ,

$$K \approx \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2}$$

b. Explain why the expansion is valid for all speeds allowed in the model.

c. A particle moves at $v = 0.1c$. Find the percentage error when its kinetic energy is approximated by $\frac{1}{2}mv^2$, giving your answer to 2 significant figures.

(8 marks)

a)

$$n = -\frac{1}{2}, y = -\frac{v^2}{c^2}$$

$$\begin{aligned} \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} &= 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{v^2}{c^2}\right)^2 + \dots \\ &= 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \dots \end{aligned}$$

$$E \approx mc^2 + \frac{1}{2}mv^2 + \frac{3mv^4}{8c^2}$$

$$K = E - mc^2$$

$$K \approx \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2}$$

b)

$$\text{Valid when } \left|\frac{v^2}{c^2}\right| < 1$$

$$0 \leq v < c \text{ means } 0 \leq \frac{v^2}{c^2} < 1$$

so it is valid for every speed in the model

c)

$$\frac{v^2}{c^2} = 0.01$$

$$\begin{aligned} \text{exact } K &= mc^2 \left(0.99^{-\frac{1}{2}} - 1\right) \\ &= 0.00503781\dots mc^2 \end{aligned}$$

$$\begin{aligned} \text{approx } K &= \frac{1}{2}m(0.1c)^2 \\ &= 0.005mc^2 \end{aligned}$$

$$\begin{aligned}\% \text{ error} &= \frac{0.00503781... - 0.005}{0.00503781...} \times 100 \\ &= 0.7506...\end{aligned}$$

0.75%