



- 1 Find the general solution of the differential equation

$$\frac{dy}{dx} = (2x + 1)y, \quad y > 0$$

giving your answer in the form $y = f(x)$.

(3 marks)

Separate the variables

$$\int \frac{1}{y} dy = \int (2x + 1) dx$$

$$\ln y = x^2 + x + c$$

Take e of both sides

$$y = e^{x^2 + x + c}$$

$$y = e^c \times e^{x^2 + x}$$

Let $A = e^c$

$$y = Ae^{x^2 + x}$$

- 2 Given that $y = 2$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = \frac{e^{2x}}{y}, \quad y > 0$$

giving your answer in the form $y = f(x)$.

(5 marks)

Separate the variables

$$\int y dy = \int e^{2x} dx$$

$$\frac{y^2}{2} = \frac{1}{2} e^{2x} + c$$

Use $x = 0, y = 2$

$$\frac{4}{2} = \frac{1}{2} + c$$

$$c = \frac{3}{2}$$

$$\frac{y^2}{2} = \frac{1}{2} e^{2x} + \frac{3}{2}$$

$$y^2 = e^{2x} + 3$$

$y > 0$, so take the positive root

$$y = \sqrt{e^{2x} + 3}$$

3

A scientist grows bacteria in a dish. The number of bacteria, N , t hours after the start of the experiment increases at a rate proportional to N . Initially there are 500 bacteria and after 3 hours there are 2000 bacteria.

- By forming and solving a differential equation, show that $N = 500e^{kt}$, where k is a positive constant.
- Find the exact value of k .
- Find, according to the model, the time taken for the number of bacteria to reach 10 000. Give your answer in hours to 3 significant figures.

(5 marks)

a) Rate proportional to N

$$\frac{dN}{dt} = kN$$

$$\int \frac{1}{N} dN = \int k dt$$

$$\ln N = kt + c$$

$$N = Ae^{kt}$$

$t = 0, N = 500$

$$A = 500$$

$$N = 500e^{kt} \text{ shown}$$

b) $t = 3, N = 2000$

$$2000 = 500e^{3k}$$

$$e^{3k} = 4$$

$$3k = \ln 4$$

$$k = \frac{1}{3} \ln 4$$

c) Set $N = 10\,000$

$$500e^{kt} = 10\,000$$

$$e^{kt} = 20$$

$$t = \frac{\ln 20}{k}$$

$$= \frac{3 \ln 20}{\ln 4}$$

$$= 6.483...$$

$$6.48 \text{ hours}$$

4

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = y \tan x, \quad 0 \leq x < \frac{\pi}{2}, \quad y > 0$$

Given that $y = 3$ when $x = 0$,

- show that $y = 3 \sec x$
- find the exact value of x for which $y = 6$

(5 marks)

a) Separate the variables

$$\int \frac{1}{y} dy = \int \tan x dx$$

$$\int \tan x dx = -\ln(\cos x) = \ln(\sec x)$$

$$\ln y = \ln(\sec x) + c$$

$$x = 0, y = 3 (\sec 0 = 1)$$

$$\ln 3 = \ln 1 + c$$

$$c = \ln 3$$

$$\ln y = \ln(\sec x) + \ln 3$$

$$\ln y = \ln(3 \sec x)$$

$$y = 3 \sec x \text{ shown}$$

b) Set $y = 6$

$$3 \sec x = 6$$

$$\sec x = 2$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

5

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = x e^{-y}$$

and $y = 0$ when $x = 2$

- Solve the differential equation, giving your answer in the form $y = \ln f(x)$
- State the set of values of x , with $x > 0$, for which your solution is defined.

(5 marks)

a) Multiply by e^y to separate

$$\int e^y dy = \int x dx$$

$$e^y = \frac{x^2}{2} + c$$

$$x = 2, y = 0$$

$$e^0 = \frac{4}{2} + c$$

$$1 = 2 + c$$

$$c = -1$$

$$e^y = \frac{x^2}{2} - 1$$

$$e^y = \frac{x^2 - 2}{2}$$

$$y = \ln\left(\frac{x^2 - 2}{2}\right)$$

b) $\ln$ needs a positive input

$$\frac{x^2 - 2}{2} > 0$$

$$x^2 > 2$$

$$x > \sqrt{2}$$

6

A young tree is planted in a park. Its height, h metres, t years after planting is modelled by the differential equation

$$\frac{dh}{dt} = \frac{6-h}{t+1}, \quad t \geq 0, \quad 1 \leq h < 6$$

The tree is 1 metre tall when it is planted.

a. Show that $h = 6 - \frac{5}{t+1}$

b. Find, according to the model, the height of the tree 4 years after planting.

c. State the height that the model predicts the tree approaches in the long term.

(6 marks)

a) Separate the variables

$$\int \frac{1}{6-h} dh = \int \frac{1}{t+1} dt$$

The $-h$ gives a minus sign

$$-\ln(6-h) = \ln(t+1) + c$$

$$t=0, h=1$$

$$-\ln 5 = \ln 1 + c$$

$$c = -\ln 5$$

$$\ln(6-h) = \ln 5 - \ln(t+1)$$

$$\ln(6-h) = \ln \frac{5}{t+1}$$

$$6-h = \frac{5}{t+1}$$

$$h = 6 - \frac{5}{t+1} \text{ shown}$$

b) $t=4$

$$h = 6 - \frac{5}{5}$$

$$5 \text{ m}$$

c) As $t \rightarrow \infty$, $\frac{5}{t+1} \rightarrow 0$

$$6 \text{ m}$$

7

A block of ice is left to melt. Its mass, m kg, t hours after it was put out is modelled as decreasing at a rate proportional to the square root of m . The initial mass is 100 kg and after 5 hours the mass is 64 kg.

- Write down a differential equation linking m and t .
- Solve your differential equation to show that $m = (10 - 0.4t)^2$
- Find, according to the model, the time taken for the ice to melt completely.

(7 marks)

a) Decreasing, so a minus sign

$$\frac{dm}{dt} = -k\sqrt{m}$$

b) Separate the variables

$$\int m^{-1/2} dm = \int -k dt$$

$$2m^{1/2} = -kt + c$$

$$t = 0, m = 100$$

$$2 \times 10 = c$$

$$c = 20$$

$$t = 5, m = 64$$

$$2 \times 8 = -5k + 20$$

$$5k = 4$$

$$k = 0.8$$

$$2\sqrt{m} = 20 - 0.8t$$

$$\sqrt{m} = 10 - 0.4t$$

$$m = (10 - 0.4t)^2 \text{ shown}$$

c) Set $m = 0$

$$10 - 0.4t = 0$$

$$t = \frac{10}{0.4}$$

$$25 \text{ hours}$$

8

A mug of tea is made in a kitchen where the room temperature is a constant 20°C . The temperature of the tea, $\theta^{\circ}\text{C}$, t minutes after it is made is modelled by

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

where k is a positive constant. The tea is made at 90°C and after 10 minutes its temperature is 60°C .

- Show that $\theta = 20 + 70e^{-kt}$
- Show that $k = \frac{1}{10} \ln \frac{7}{4}$
- Find, according to the model, the time taken for the tea to cool to 30°C . Give your answer to the nearest minute.
- State what happens to the temperature of the tea in the long term, according to the model.

(8 marks)

a) Separate the variables

$$\int \frac{1}{\theta - 20} d\theta = \int -k dt$$

$$\ln(\theta - 20) = -kt + c$$

$$\theta - 20 = Ae^{-kt}$$

$$t = 0, \theta = 90$$

$$70 = A$$

$$\theta = 20 + 70e^{-kt} \text{ shown}$$

$$b) t = 10, \theta = 60$$

$$60 = 20 + 70e^{-10k}$$

$$e^{-10k} = \frac{40}{70} = \frac{4}{7}$$

$$-10k = \ln \frac{4}{7}$$

$$k = \frac{1}{10} \ln \frac{7}{4}$$

shown

$$c) \text{ Set } \theta = 30$$

$$30 = 20 + 70e^{-kt}$$

$$e^{-kt} = \frac{1}{7}$$

$$kt = \ln 7$$

$$t = \frac{\ln 7}{k}$$

$$= \frac{10 \ln 7}{\ln 1.75}$$

$$= 34.77...$$

35 minutes

d) As $t \rightarrow \infty$, $e^{-kt} \rightarrow 0$

$\theta \rightarrow 20^\circ \text{C}$ (room temperature)

9

Water is pumped into a garden pond at a constant rate of 0.6 m^3 per hour. At the same time water leaves the pond through an overflow pipe at a rate of $0.2V \text{ m}^3$ per hour, where $V \text{ m}^3$ is the volume of water in the pond t hours after the pump is switched on.

a. Show that $5 \frac{dV}{dt} = 3 - V$

b. Given that $V = 1$ when $t = 0$, show that $V = 3 - 2e^{-0.2t}$

c. Find, according to the model, the time taken for the volume to reach 2.5 m^3 . Give your answer in hours to 3 significant figures.

d. Explain why, according to the model, the volume of water in the pond can never reach 3 m^3 .

(8 marks)

a) Rate of change = rate in - rate out

$$\frac{dV}{dt} = 0.6 - 0.2V$$

Multiply by 5

$$5 \frac{dV}{dt} = 3 - V \text{ shown}$$

b) Separate the variables

$$\int \frac{5}{3 - V} dV = \int 1 dt$$

$$-5 \ln(3 - V) = t + c$$

$t = 0, V = 1$

$$-5 \ln 2 = c$$

$$-5 \ln(3 - V) = t - 5 \ln 2$$

$$\ln(3 - V) = \ln 2 - 0.2t$$

$$3 - V = e^{\ln 2 - 0.2t}$$

$$3 - V = 2e^{-0.2t}$$

$$V = 3 - 2e^{-0.2t} \text{ shown}$$

c) Set $V = 2.5$

$$2e^{-0.2t} = 0.5$$

$$e^{-0.2t} = 0.25$$

$$-0.2t = \ln 0.25$$

$$t = 5 \ln 4$$

$$= 6.931...$$

$$6.93 \text{ hours}$$

d) $2e^{-0.2t} > 0$ for every t

$$V = 3 - (\text{a positive amount})$$

$$V < 3 \text{ always, so never } 3 \text{ m}^3$$

10

After a storm a puddle on a flat playground is modelled as a circle. At time t hours the radius of the puddle is r cm and its area is A cm². In the model the area decreases at a rate inversely proportional to r .

a. Show that $\frac{dr}{dt} = -\frac{c}{r^2}$, where c is a positive constant.

b. Given that the radius is 30 cm at $t = 0$ and 20 cm at $t = 1$, show that $r^3 = 27\,000 - 19\,000t$

c. Find, according to the model, the time taken for the puddle to dry up completely. Give your answer to the nearest minute.

(9 marks)

a) Inversely proportional to r , and decreasing

$$\frac{dA}{dt} = -\frac{k}{r}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

Chain rule

$$\begin{aligned}\frac{dr}{dt} &= \frac{dA}{dt} \div \frac{dA}{dr} \\ &= -\frac{k}{r} \times \frac{1}{2\pi r} \\ &= -\frac{k}{2\pi r^2}\end{aligned}$$

Let $c = \frac{k}{2\pi}$, a positive constant

$$\boxed{\frac{dr}{dt} = -\frac{c}{r^2} \text{ shown}}$$

b) Separate the variables

$$\begin{aligned}\int r^2 dr &= \int -c dt \\ \frac{r^3}{3} &= -ct + d\end{aligned}$$

$t = 0, r = 30$

$$d = \frac{27\,000}{3} = 9000$$

$t = 1, r = 20$

$$\begin{aligned}\frac{8000}{3} &= -c + 9000 \\ c &= \frac{19\,000}{3}\end{aligned}$$

$$\frac{r^3}{3} = 9000 - \frac{19000}{3}t$$

Multiply by 3

$$r^3 = 27000 - 19000t \text{ shown}$$

c) Dry when $r = 0$

$$27000 - 19000t = 0$$

$$t = \frac{27}{19} \text{ hours}$$

$$= 1.421... \times 60 = 85.26 \text{ minutes}$$

85 minutes

11

The number of fish, P hundred, in a lake t years after a group of fish was released into it is modelled by the differential equation

$$\frac{dP}{dt} = \frac{P(10 - P)}{20}, \quad 0 < P < 10$$

The group released contained 200 fish.

a. Express $\frac{1}{P(10 - P)}$ in partial fractions.

b. Show that $P = \frac{10}{1 + 4e^{-0.5t}}$

c. Find, according to the model, the time taken for the number of fish to reach 800. Give your answer in the form $a \ln 2$ years, where a is an integer.

d. State the maximum number of fish the lake can support, according to the model.

(10 marks)

a) Partial fractions

$$\frac{1}{P(10 - P)} \equiv \frac{A}{P} + \frac{B}{10 - P}$$

$$1 \equiv A(10 - P) + BP$$

$$P = 0$$

$$A = \frac{1}{10}$$

$$P = 10$$

$$B = \frac{1}{10}$$

$$\boxed{\frac{1}{10P} + \frac{1}{10(10 - P)}}$$

b) Separate the variables (200 fish means $P = 2$ at $t = 0$)

$$\int \frac{20}{P(10 - P)} dP = \int 1 dt$$

$$\int \left(\frac{2}{P} + \frac{2}{10 - P} \right) dP = \int 1 dt$$

$$2 \ln P - 2 \ln(10 - P) = t + c$$

$$t = 0, P = 2$$

$$2 \ln 2 - 2 \ln 8 = c$$

$$c = 2 \ln \frac{1}{4}$$

$$2 \ln \frac{P}{10 - P} - 2 \ln \frac{1}{4} = t$$

$$\ln \frac{4P}{10 - P} = \frac{t}{2}$$

$$\frac{4P}{10 - P} = e^{0.5t}$$

$$4P = 10e^{0.5t} - Pe^{0.5t}$$

$$P(4 + e^{0.5t}) = 10e^{0.5t}$$

Divide top and bottom by $e^{0.5t}$

$$P = \frac{10}{1 + 4e^{-0.5t}} \text{ shown}$$

c) 800 fish means $P = 8$

$$8 = \frac{10}{1 + 4e^{-0.5t}}$$

$$1 + 4e^{-0.5t} = 1.25$$

$$e^{-0.5t} = \frac{1}{16}$$

$$0.5t = \ln 16$$

$$t = 2 \ln 16 = 8 \ln 2$$

$$8 \ln 2 \text{ years}$$

d) As $t \rightarrow \infty$, $e^{-0.5t} \rightarrow 0$ so $P \rightarrow 10$

$$1000 \text{ fish}$$

A gardener measures a seedling. Its height, h cm, t weeks after the first measurement is modelled by the differential equation

$$\frac{dh}{dt} = t e^{-0.5t} \sqrt{h}, \quad t \geq 0$$

When $t = 0$ the seedling is 4 cm tall.

- Show that $\sqrt{h} = 4 - (t + 2)e^{-0.5t}$
- Find, according to the model, the height of the seedling 4 weeks after the first measurement. Give your answer to 3 significant figures.
- Find the height that the model predicts the seedling approaches in the long term, and give one reason why the model may not be realistic for a real plant.

(10 marks)

a) Separate the variables

$$\int h^{-1/2} dh = \int t e^{-0.5t} dt$$

$$\int h^{-1/2} dh = 2h^{1/2}$$

Right side by parts

$$u = t$$

$$\frac{du}{dt} = 1$$

$$\frac{dv}{dt} = e^{-0.5t}$$

$$v = -2e^{-0.5t}$$

$$\int t e^{-0.5t} dt = -2t e^{-0.5t} - \int -2e^{-0.5t} dt$$

$$= -2t e^{-0.5t} - 4e^{-0.5t}$$

$$2\sqrt{h} = -2t e^{-0.5t} - 4e^{-0.5t} + c$$

$$t = 0, h = 4$$

$$2 \times 2 = 0 - 4 + c$$

$$c = 8$$

$$2\sqrt{h} = 8 - (2t + 4)e^{-0.5t}$$

Divide by 2

$$\sqrt{h} = 4 - (t + 2)e^{-0.5t} \text{ shown}$$

$$b) t = 4$$

$$\sqrt{h} = 4 - 6e^{-2}$$

$$= 3.1879...$$

$$h = 3.1879...^2 = 10.163...$$

$$10.2 \text{ cm}$$

c) As $t \rightarrow \infty$, $(t+2)e^{-0.5t} \rightarrow 0$

$$\int h \rightarrow 4$$

$$h \rightarrow 16$$

16 cm

Reason: the model stops growth at 16 cm, but a real plant's growth depends on light, water and the season