



1 Find $\frac{dy}{dx}$ for each of the following.

a. $y = (4x^3 + 1)^5$

b. $y = \ln(5x^2 + 2)$

c. $y = \sec 4x$

(5 marks)

a)

Chain rule: power down, reduce the power, times the derivative of the inside

$$\frac{dy}{dx} = 5(4x^3 + 1)^4 \times 12x^2$$

$$60x^2(4x^3 + 1)^4$$

b)

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

$$\frac{dy}{dx} = \frac{10x}{5x^2 + 2}$$

c)

$$\frac{d}{dx} \sec x = \sec x \tan x, \text{ then times the derivative of } 4x$$

$$\frac{dy}{dx} = 4 \sec 4x \tan 4x$$

2

a. Given $y = x^3 e^{2x}$, show that $\frac{dy}{dx} = x^2 e^{2x}(a + bx)$, where a and b are integers to be found.

b. Given $y = \frac{3x-1}{x^2+2}$, find $\frac{dy}{dx}$, giving your answer as a single fraction in its simplest form.

(5 marks)

a)

Product rule: $u = x^3, v = e^{2x}$

$$u' = 3x^2$$

$$v' = 2e^{2x}$$

$$\frac{dy}{dx} = u'v + uv'$$

$$= 3x^2 e^{2x} + 2x^3 e^{2x}$$

$$= x^2 e^{2x}(3 + 2x)$$

$$\boxed{a = 3, b = 2}$$

b)

Quotient rule: $u = 3x - 1, v = x^2 + 2$

$$u' = 3$$

$$v' = 2x$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{3(x^2 + 2) - 2x(3x - 1)}{(x^2 + 2)^2}$$

$$= \frac{3x^2 + 6 - 6x^2 + 2x}{(x^2 + 2)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2 + 2x + 6}{(x^2 + 2)^2}}$$

3

a. The curve with equation $y = 3^{2x}$ passes through the point P where $x = 1$. Find the exact gradient of the curve at P , giving your answer in the form $k \ln 3$

b. Given that $x = \cos y$, where $0 < y < \pi$, show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(6 marks)

a)

$$y = 3^{2x} = 9^x$$

$$\frac{d}{dx} a^x = a^x \ln a$$

$$\frac{dy}{dx} = 9^x \ln 9$$

At $x = 1$

$$\frac{dy}{dx} = 9 \ln 9$$

$$= 9 \times 2 \ln 3$$

$$\boxed{18 \ln 3}$$

b)

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = 1 \div \frac{dx}{dy}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\sin^2 y = 1 - \cos^2 y = 1 - x^2$$

$\sin y > 0$ for $0 < y < \pi$, so take the positive root

$$\sin y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\text{shown}}$$

4

Given that x is measured in radians, prove, from first principles, that the derivative of $\sin 3x$ is $3 \cos 3x$

You may assume the formula for $\sin(A \pm B)$ and that, as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$

(5 marks)

Gradient of the chord from x to $x + h$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(3x+3h) - \sin 3x}{h}$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{\sin 3x \cos 3h + \cos 3x \sin 3h - \sin 3x}{h}$$

$$= \sin 3x \left(\frac{\cos 3h - 1}{h} \right) + \cos 3x \left(\frac{\sin 3h}{h} \right)$$

Make the limits match: multiply top and bottom by 3

$$= 3 \sin 3x \left(\frac{\cos 3h - 1}{3h} \right) + 3 \cos 3x \left(\frac{\sin 3h}{3h} \right)$$

As $h \rightarrow 0$, $3h \rightarrow 0$

$$\frac{\cos 3h - 1}{3h} \rightarrow 0$$

$$\frac{\sin 3h}{3h} \rightarrow 1$$

$$\begin{aligned} f'(x) &= 3 \sin 3x \times 0 + 3 \cos 3x \times 1 \\ &= 3 \cos 3x \end{aligned}$$

proved

5

The curve C has equation $y = (x^2 - 3)e^x$

- Find the exact coordinates of the stationary points of C .
- Use $\frac{d^2y}{dx^2}$ to determine the nature of each stationary point.

(7 marks)

a)

Product rule: $u = x^2 - 3$, $v = e^x$

$$\begin{aligned}\frac{dy}{dx} &= 2x e^x + (x^2 - 3)e^x \\ &= e^x(x^2 + 2x - 3) \\ &= e^x(x + 3)(x - 1)\end{aligned}$$

$e^x > 0$, so the bracket must be 0

$$x = 1$$

$$x = -3$$

$$y = (1 - 3)e^1 = -2e$$

$$y = (9 - 3)e^{-3} = 6e^{-3}$$

$$(1, -2e) \text{ and } (-3, 6e^{-3})$$

b)

Product rule again on $e^x(x^2 + 2x - 3)$

$$\begin{aligned}\frac{d^2y}{dx^2} &= e^x(x^2 + 2x - 3) + e^x(2x + 2) \\ &= e^x(x^2 + 4x - 1)\end{aligned}$$

$$x = 1$$

$$\frac{d^2y}{dx^2} = 4e > 0$$

$$x = -3$$

$$\frac{d^2y}{dx^2} = e^{-3}(9 - 12 - 1) = -4e^{-3} < 0$$

$$(1, -2e) \text{ minimum, } (-3, 6e^{-3}) \text{ maximum}$$

6

The curve C has equation

$$x^2 + 3xy - y^2 = 27$$

a. Find $\frac{dy}{dx}$ in terms of x and y .b. The point $P(3, 3)$ lies on C . Find an equation of the normal to C at P , giving your answer in the form $ax + by + c = 0$, where a , b and c are integers.

(6 marks)

a)

Product rule on $3xy$; chain rule on y^2

$$2x + 3y + 3x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x - 2y) = -(2x + 3y)$$

$$\boxed{\frac{dy}{dx} = -\frac{2x + 3y}{3x - 2y}}$$

b)

$$\frac{dy}{dx} = -\frac{6 + 9}{9 - 6}$$

$$= -5$$

Normal gradient = $-1 \div (-5)$

$$m = \frac{1}{5}$$

$$y - 3 = \frac{1}{5}(x - 3)$$

$$5y - 15 = x - 3$$

$$\boxed{x - 5y + 12 = 0}$$

7

The curve C has equation

$$y = \frac{\sin x}{1 + \cos x}, \quad 0 < x < \pi$$

a. Show that $\frac{dy}{dx} = \frac{1}{1 + \cos x}$

b. Find the exact x coordinate of the point on C where the gradient is $\frac{2}{3}$

(5 marks)

a)

Quotient rule: $u = \sin x, v = 1 + \cos x$

$$u' = \cos x$$

$$v' = -\sin x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \end{aligned}$$

$$\sin^2 x + \cos^2 x = 1$$

$$= \frac{1 + \cos x}{(1 + \cos x)^2}$$

$$= \frac{1}{1 + \cos x}$$

shown

b)

$$\frac{1}{1 + \cos x} = \frac{2}{3}$$

$$1 + \cos x = \frac{3}{2}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

8

$$f(x) = x^4 - 6x^3 + 12x^2 + 5$$

- Find $f''(x)$
- Find the set of values of x for which $f(x)$ is concave.
- Find the coordinates of the points of inflection on the curve $y = f(x)$, and show that neither of them is a stationary point.

(6 marks)

a)

$$f'(x) = 4x^3 - 18x^2 + 24x$$

$$f''(x) = 12x^2 - 36x + 24$$

b)

Concave where $f''(x) < 0$

$$12(x^2 - 3x + 2) < 0$$

$$12(x - 1)(x - 2) < 0$$

Positive quadratic: negative between the roots

$$\{x : 1 < x < 2\}$$

c)

 $f''(x) = 0$ at $x = 1$ and $x = 2$, and f'' changes sign at each

$$f(1) = 1 - 6 + 12 + 5 = 12$$

$$f(2) = 16 - 48 + 48 + 5 = 21$$

Check the gradient there

$$f'(1) = 4 - 18 + 24 = 10 \neq 0$$

$$f'(2) = 32 - 72 + 48 = 8 \neq 0$$

$$(1, 12) \text{ and } (2, 21), \text{ neither stationary}$$

9

Water is poured into a container in the shape of an inverted right circular cone at a constant rate of $40 \text{ cm}^3 \text{ s}^{-1}$. When the depth of water is $h \text{ cm}$, the radius of the water surface is $\frac{1}{2}h \text{ cm}$.

- Show that the volume of water, $V \text{ cm}^3$, is given by $V = \frac{\pi h^3}{12}$
- Find the rate at which the depth of the water is increasing at the instant when $h = 8$, giving your answer to 3 significant figures.
- Find the exact rate at which the area of the water surface is increasing at this instant.

(7 marks)

a)

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h \\ &= \frac{\pi h^3}{12} \end{aligned}$$

shown

b)

$$\frac{dV}{dh} = \frac{3\pi h^2}{12} = \frac{\pi h^2}{4}$$

At $h = 8$

$$\frac{dV}{dh} = 16\pi$$

Chain rule: $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

$$\begin{aligned} \frac{dh}{dt} &= \frac{40}{16\pi} \\ &= \frac{5}{2\pi} \end{aligned}$$

$$= 0.79577\dots$$

$$0.796 \text{ cm s}^{-1}$$

c)

$$S = \pi \left(\frac{h}{2}\right)^2 = \frac{\pi h^2}{4}$$

$$\begin{aligned} \frac{dS}{dh} &= \frac{\pi h}{2} \\ &= 4\pi \end{aligned}$$

Chain rule: $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$

$$\frac{dS}{dt} = 4\pi \times \frac{5}{2\pi}$$

$$\boxed{10 \text{ cm}^2 \text{ s}^{-1}}$$

10

The curve C has equation

$$x^2 + xy + y^2 = 12$$

a. Find $\frac{dy}{dx}$ in terms of x and y .b. Find the coordinates of the points on C where the tangent is parallel to the x -axis.c. Find the coordinates of the points on C where the tangent is parallel to the y -axis.

(8 marks)

a)

Product rule on xy ; chain rule on y^2

$$2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + 2y) = -(2x + y)$$

$$\boxed{\frac{dy}{dx} = -\frac{2x + y}{x + 2y}}$$

b)

 $\frac{dy}{dx} = 0$: numerator is 0

$$2x + y = 0$$

$$y = -2x$$

Substitute into the curve

$$x^2 + x(-2x) + (-2x)^2 = 12$$

$$3x^2 = 12$$

$$x = \pm 2$$

$$\boxed{(2, -4) \text{ and } (-2, 4)}$$

c)

Parallel to the y -axis: denominator is 0

$$x + 2y = 0$$

$$x = -2y$$

$$(-2y)^2 + (-2y)y + y^2 = 12$$

$$3y^2 = 12$$

$$y = \pm 2$$

$$\boxed{(-4, 2) \text{ and } (4, -2)}$$

11

The number of people, N thousand, who have downloaded a new revision app t weeks after its launch is modelled by

$$N = \frac{40}{1 + 7e^{-0.5t}}, \quad t \geq 0$$

- State the number of downloads at launch and the long-term limit of the number of downloads according to the model.
- Show that $\frac{dN}{dt} = \frac{N(40 - N)}{80}$
- Hence find the greatest rate of downloads, in thousands per week, according to the model, and the value of t at which it occurs. Give t to 3 significant figures.
- Explain why the graph of N against t has a point of inflection at this value of t .

(9 marks)

a)

$$t = 0: N = \frac{40}{1 + 7} = 5$$

As $t \rightarrow \infty$, $e^{-0.5t} \rightarrow 0$

$$N \rightarrow \frac{40}{1} = 40$$

5000 at launch; limit 40 000

b)

$$N = 40(1 + 7e^{-0.5t})^{-1}$$

Chain rule

$$\begin{aligned} \frac{dN}{dt} &= -40(1 + 7e^{-0.5t})^{-2} \times (-3.5e^{-0.5t}) \\ &= \frac{140e^{-0.5t}}{(1 + 7e^{-0.5t})^2} \end{aligned}$$

Now work out the right-hand side

$$\begin{aligned} 40 - N &= \frac{40(1 + 7e^{-0.5t}) - 40}{1 + 7e^{-0.5t}} \\ &= \frac{280e^{-0.5t}}{1 + 7e^{-0.5t}} \\ \frac{N(40 - N)}{80} &= \frac{1}{80} \times \frac{40}{1 + 7e^{-0.5t}} \times \frac{280e^{-0.5t}}{1 + 7e^{-0.5t}} \\ &= \frac{140e^{-0.5t}}{(1 + 7e^{-0.5t})^2} \end{aligned}$$

shown

c)

Rate = $\frac{1}{80}(40N - N^2)$: a quadratic in N , greatest at $N = 20$ (by symmetry)

$$\frac{dN}{dt} = \frac{20 \times 20}{80}$$

$$= 5$$

$$1 + \frac{40}{7e^{-0.5t}} = 20$$

$$1 + 7e^{-0.5t} = 2$$

$$e^{-0.5t} = \frac{1}{7}$$

$$-0.5t = \ln \frac{1}{7}$$

$$t = 2 \ln 7$$

$$= 3.8918...$$

5 thousand per week at $t = 3.89$

d)

$\frac{dN}{dt}$ increases while $N < 20$ and decreases once $N > 20$

so $\frac{d^2N}{dt^2}$ changes sign from + to - at $t = 2 \ln 7$

convex changes to concave: a point of inflection

12 The curve C has equation $y = f(x)$, where

$$f(x) = \frac{\ln x}{x^2}, \quad x > 0$$

a. Show that $f'(x) = \frac{1 - 2 \ln x}{x^3}$

b. Find the exact coordinates of the stationary point of C , and use $f''(x)$ to show that it is a maximum.

c. Find the set of values of x for which C is convex, giving the boundary in the form e^k

d. Given that $f(x) \rightarrow 0$ as $x \rightarrow \infty$, state, with a reason, the number of solutions of $f(x) = k$ for $0 < k < \frac{1}{2e}$

(10 marks)

a)

Quotient rule: $u = \ln x, v = x^2$

$$u' = \frac{1}{x}$$

$$v' = 2x$$

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x} \times x^2 - 2x \ln x}{x^4} \\ &= \frac{x - 2x \ln x}{x^4} \\ &= \frac{1 - 2 \ln x}{x^3} \end{aligned}$$

shown

b)

$$1 - 2 \ln x = 0$$

$$\ln x = \frac{1}{2}$$

$$x = e^{1/2} = \sqrt{e}$$

$$y = \frac{\frac{1}{2}}{e} = \frac{1}{2e}$$

Quotient rule on $f'(x)$: $u = 1 - 2 \ln x, v = x^3$

$$\begin{aligned} f''(x) &= \frac{-\frac{2}{x} \times x^3 - 3x^2(1 - 2 \ln x)}{x^6} \\ &= \frac{-2x^2 - 3x^2 + 6x^2 \ln x}{x^6} \end{aligned}$$

$$= \frac{6 \ln x - 5}{x^4}$$

At $x = \sqrt{e}$, $\ln x = \frac{1}{2}$

$$f''(\sqrt{e}) = \frac{3 - 5}{e^2} = -\frac{2}{e^2} < 0$$

$$\left(\sqrt{e}, \frac{1}{2e}\right), \text{ a maximum}$$

c)

Convex where $f''(x) > 0$, and $x^4 > 0$

$$6 \ln x - 5 > 0$$

$$\ln x > \frac{5}{6}$$

$$\{x : x > e^{5/6}\}$$

d)

As $x \rightarrow 0$, $f(x) \rightarrow -\infty$; $f(1) = 0$

f rises to $\frac{1}{2e}$ at $x = \sqrt{e}$, then falls towards 0 but stays positive

so $y = k$ meets C once on each side of the maximum

two solutions