



- 1 The functions f and g are defined by

$$f(x) = 3x - 5, x \in \mathbb{R}$$

$$g(x) = x^2 + 1, x \in \mathbb{R}$$

- a. Find $fg(2)$
b. Find $gf(x)$, giving your answer in the form $ax^2 + bx + c$

(4 marks)

a)

Work from the inside out: g first, then f

$$\begin{aligned} g(2) &= 2^2 + 1 \\ &= 5 \end{aligned}$$

$$\begin{aligned} fg(2) &= f(5) \\ &= 3 \times 5 - 5 \end{aligned}$$

$$\boxed{fg(2) = 10}$$

b)

f goes in first, so put $3x - 5$ into g

$$\begin{aligned} gf(x) &= g(3x - 5) \\ &= (3x - 5)^2 + 1 \\ &= 9x^2 - 30x + 25 + 1 \end{aligned}$$

$$\boxed{gf(x) = 9x^2 - 30x + 26}$$

2

The function f is defined by

$$f(x) = \frac{2x+3}{x-1}, x > 1$$

a. Find $f^{-1}(x)$ b. State the domain of f^{-1}

(4 marks)

a)

Write $y = f(x)$ and make x the subject

$$y = \frac{2x+3}{x-1}$$

$$y(x-1) = 2x+3$$

$$xy - y = 2x + 3$$

$$xy - 2x = y + 3$$

$$x(y-2) = y+3$$

$$x = \frac{y+3}{y-2}$$

$$f^{-1}(x) = \frac{x+3}{x-2}$$

b)

Domain of f^{-1} = range of f

$$f(x) = \frac{2(x-1)+5}{x-1}$$

$$= 2 + \frac{5}{x-1}$$

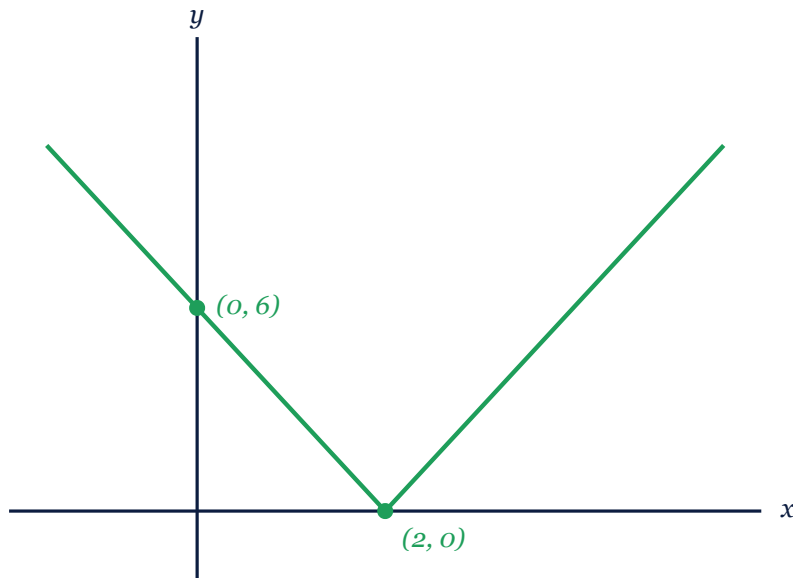
For $x > 1$, $\frac{5}{x-1}$ takes every positive value

$$f(x) > 2$$

$$\text{domain of } f^{-1}: x > 2$$

3

- a. Sketch the graph of $y = |3x - 6|$, showing the coordinates of any points where the graph meets the coordinate axes.
- b. Solve the equation $|3x - 6| = 9$



(5 marks)

a)

The V touches the x-axis where $3x - 6 = 0$

$$3x - 6 = 0$$

$$x = 2$$

y-intercept: put $x = 0$

$$y = |0 - 6|$$

$$= 6$$

vertex (2, 0), y-intercept (0, 6)

b)

Take the positive and the negative case

$$3x - 6 = 9$$

$$3x = 15$$

$$x = 5$$

$$3x - 6 = -9$$

$$3x = -3$$

$$x = -1$$

$$x = -1 \text{ or } x = 5$$

4

The function f is defined by

$$f(x) = x^2 - 6x + 11, x \geq 3$$

a. Find the range of f .b. Find $f^{-1}(x)$, stating its domain.c. Solve the equation $f(x) = f^{-1}(x)$, giving your answer in the form $\frac{a + \sqrt{b}}{2}$

(7 marks)

a)

Complete the square

$$\begin{aligned} f(x) &= (x - 3)^2 - 9 + 11 \\ &= (x - 3)^2 + 2 \end{aligned}$$

Least value when $x = 3$

$$f(x) \geq 2$$

b)

$$\begin{aligned} y &= (x - 3)^2 + 2 \\ (x - 3)^2 &= y - 2 \end{aligned}$$

 $x \geq 3$ so take the positive root

$$\begin{aligned} x - 3 &= \sqrt{y - 2} \\ x &= 3 + \sqrt{y - 2} \end{aligned}$$

$$f^{-1}(x) = 3 + \sqrt{x - 2}, x \geq 2$$

c)

 f is increasing, so f and f^{-1} meet on the line $y = x$

$$x^2 - 6x + 11 = x$$

$$x^2 - 7x + 11 = 0$$

$$x = \frac{7 \pm \sqrt{49 - 44}}{2}$$

$$x = \frac{7 \pm \sqrt{5}}{2}$$

 $\frac{7 - \sqrt{5}}{2} = 2.38\dots$ is less than 3, so reject it

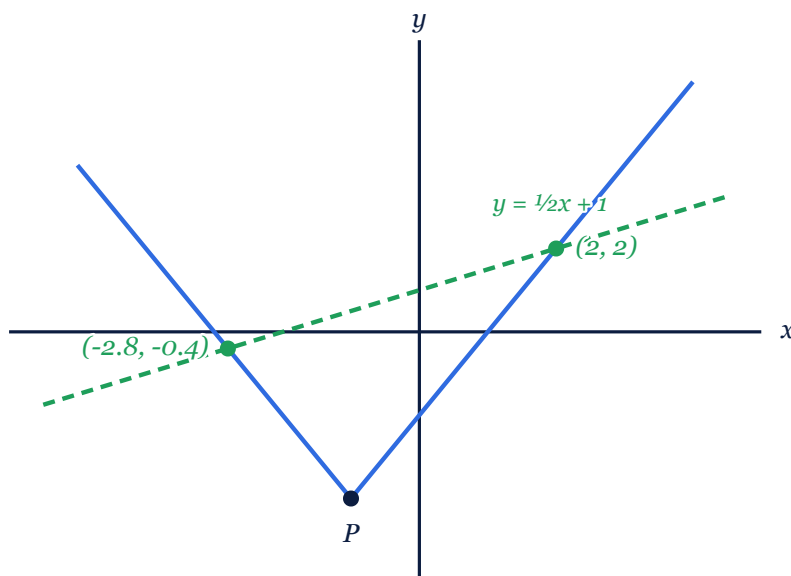
$$x = \frac{7 + \sqrt{5}}{2}$$

5

The diagram shows a sketch of the graph with equation $y = 2|x + 1| - 4$

The vertex of the graph is the point P.

- Find the coordinates of P.
- Find the coordinates of the points where the graph meets the x -axis.
- Solve the equation $2|x + 1| - 4 = \frac{1}{2}x + 1$
- Find the set of values of the constant k for which the equation $2|x + 1| - 4 = k$ has exactly two solutions.



(8 marks)

a)

Vertex where the inside of the modulus is 0

$$x + 1 = 0$$

$$x = -1$$

$$y = 2(0) - 4$$

$$\boxed{P(-1, -4)}$$

b)

$$2|x + 1| - 4 = 0$$

$$|x + 1| = 2$$

$$x + 1 = 2 \text{ or } x + 1 = -2$$

$$\boxed{(1, 0) \text{ and } (-3, 0)}$$

c)

Right branch, $x \geq -1$: $|x + 1| = x + 1$

$$2(x + 1) - 4 = \frac{1}{2}x + 1$$

$$2x - 2 = \frac{1}{2}x + 1$$

$$\frac{3}{2}x = 3$$

$$x = 2$$

$x = 2 \geq -1$ so it is valid

Left branch, $x < -1$: $|x + 1| = -(x + 1)$

$$-2(x + 1) - 4 = \frac{1}{2}x + 1$$

$$-2x - 6 = \frac{1}{2}x + 1$$

$$-\frac{5}{2}x = 7$$

$$x = -2.8$$

$x = -2.8 < -1$ so it is valid

$$x = -2.8 \text{ or } x = 2$$

d)

$y = k$ must cut both arms of the V: above the vertex

$$k > -4$$

6

The functions f and g are defined by

$$f(x) = e^{2x} - 3, x \in \mathbb{R}$$

$$g(x) = \ln(x + 4), x > -4$$

- State the range of f .
- Find $f^{-1}(x)$, stating its domain.
- Find $gf(x)$, giving your answer in its simplest form.
- Solve $gf(x) = 2$, giving your answer in the form $\frac{1}{2} \ln a$, where a is an exact value to be found.

(7 marks)

a)

 $e^{2x} > 0$ for every x

$$f(x) > -3$$

b)

$$y = e^{2x} - 3$$

$$e^{2x} = y + 3$$

$$2x = \ln(y + 3)$$

$$x = \frac{1}{2} \ln(y + 3)$$

$$f^{-1}(x) = \frac{1}{2} \ln(x + 3), x > -3$$

c)

$$\begin{aligned} gf(x) &= g(e^{2x} - 3) \\ &= \ln(e^{2x} - 3 + 4) \end{aligned}$$

$$gf(x) = \ln(e^{2x} + 1)$$

d)

$$\ln(e^{2x} + 1) = 2$$

Take e to the power of both sides

$$e^{2x} + 1 = e^2$$

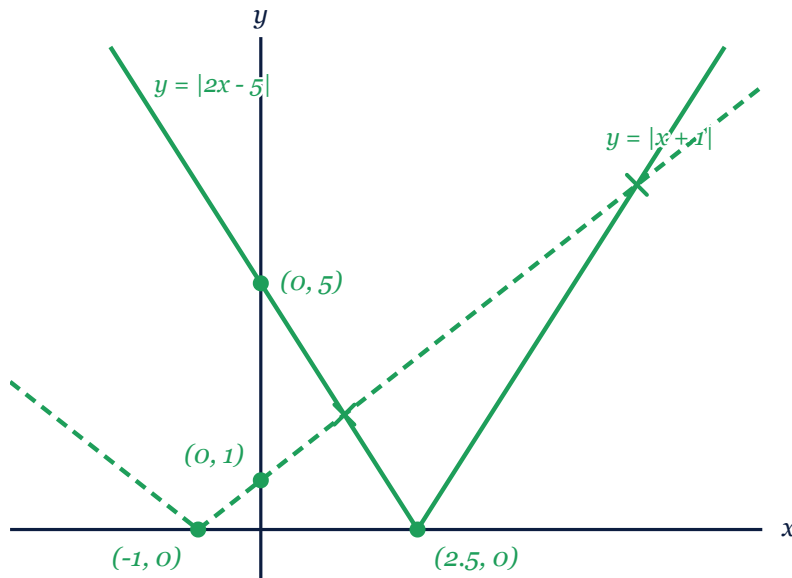
$$e^{2x} = e^2 - 1$$

$$2x = \ln(e^2 - 1)$$

$$x = \frac{1}{2} \ln(e^2 - 1)$$

7

- a. On the same axes, sketch the graphs of $y = |2x - 5|$ and $y = |x + 1|$, showing the coordinates of the points where each graph meets the coordinate axes.
- b. Solve the inequality $|2x - 5| > |x + 1|$



(7 marks)

a)

$y = |2x - 5|$: vertex where $2x - 5 = 0$

$(2.5, 0)$ and $(0, 5)$

$y = |x + 1|$: vertex where $x + 1 = 0$

$(-1, 0)$ and $(0, 1)$

b)

Both sides are ≥ 0 , so squaring keeps the inequality

$$(2x - 5)^2 > (x + 1)^2$$

$$4x^2 - 20x + 25 > x^2 + 2x + 1$$

$$3x^2 - 22x + 24 > 0$$

$$(3x - 4)(x - 6) > 0$$

Critical values

$$x = \frac{4}{3} \text{ or } x = 6$$

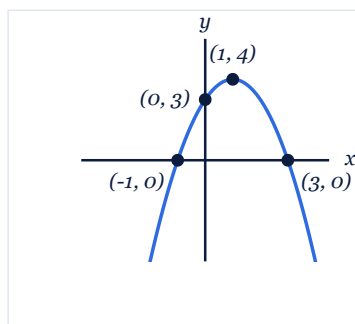
Positive quadratic: outside the roots

$$x < \frac{4}{3} \text{ or } x > 6$$

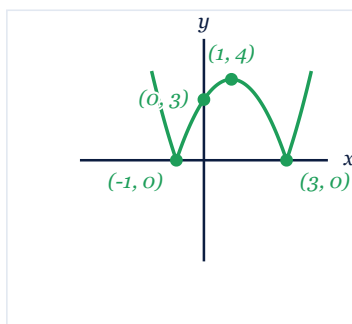
8

The diagram shows a sketch of the curve with equation $y = f(x)$. The curve meets the x -axis at $(-1, 0)$ and $(3, 0)$, meets the y -axis at $(0, 3)$ and has a maximum point at $(1, 4)$.

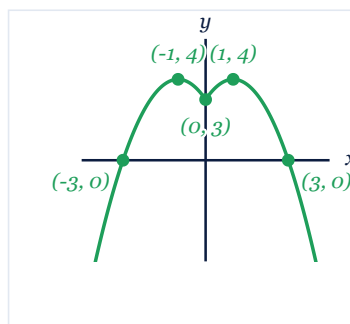
- Sketch the curve with equation $y = |f(x)|$, showing the coordinates of the points where it meets the axes and of the maximum point.
- Sketch the curve with equation $y = f(|x|)$, showing the coordinates of the points where it meets the axes and of any maximum points.
- Find the set of values of k for which $f(|x|) = k$ has exactly four solutions.
- State the coordinates of the maximum point of the curve with equation $y = 2f(x + 1)$



$y = f(x)$



(a) $y = |f(x)|$



(b) $y = f(|x|)$

(8 marks)

a)

$|f(x)|$: reflect the parts below the x -axis upwards

meets axes at $(-1, 0)$, $(3, 0)$, $(0, 3)$; maximum $(1, 4)$

b)

$f(|x|)$: keep $x \geq 0$, reflect it in the y -axis

maxima $(-1, 4)$ and $(1, 4)$; $(-3, 0)$, $(3, 0)$, $(0, 3)$

c)

$y = k$ cuts both humps twice when it is between the dip and the tops

The dip is at $(0, 3)$ and the tops are at $y = 4$

$$3 < k < 4$$

d)

$x + 1$: translate 1 to the left

$$(1, 4) \rightarrow (0, 4)$$

$2f$: stretch parallel to the y -axis, scale factor 2

$$(0, 4) \rightarrow (0, 8)$$

$$(0, 8)$$

9

The function h is defined by

$$h(x) = 5 + 8x - 2x^2, x \in \mathbb{R}$$

- Explain why h does not have an inverse.
- Write $h(x)$ in the form $p - q(x - s)^2$, where p , q and s are constants.

The function g is defined by $g(x) = 5 + 8x - 2x^2, x \leq k$

- State the largest value of k for which g has an inverse.
- Using this value of k , find $g^{-1}(x)$, stating its domain.

(7 marks)

a)

$$h(0) = 5$$

$$h(4) = 5 + 32 - 32 = 5$$

h is many-to-one, so it has no inverse

b)

Complete the square

$$\begin{aligned} h(x) &= 5 - 2(x^2 - 4x) \\ &= 5 - 2[(x - 2)^2 - 4] \end{aligned}$$

$$h(x) = 13 - 2(x - 2)^2$$

c)

Maximum at $x = 2$: h is one-to-one to the left of it

$$k = 2$$

d)

$$\begin{aligned} y &= 13 - 2(x - 2)^2 \\ 2(x - 2)^2 &= 13 - y \\ (x - 2)^2 &= \frac{13 - y}{2} \end{aligned}$$

$x \leq 2$ so $x - 2 \leq 0$: take the negative root

$$\begin{aligned} x - 2 &= -\sqrt{\frac{13 - y}{2}} \\ x &= 2 - \sqrt{\frac{13 - y}{2}} \end{aligned}$$

Domain of g^{-1} = range of g

$$g^{-1}(x) = 2 - \sqrt{\frac{13 - x}{2}}, x \leq 13$$

10

The functions f and g are defined by

$$f(x) = 2x + 3, x \in \mathbb{R}$$

$$g(x) = \frac{6}{x+1}, x > -1$$

a. Show that $gf(x) = \frac{3}{x+2}$

b. Explain why $gf(x)$ is only defined for $x > -2$, and state the range of gf .

c. Show that the equation $fg(x) = gf(x)$ has no solutions.

(8 marks)

a)

$$\begin{aligned} gf(x) &= g(2x+3) \\ &= \frac{6}{2x+3+1} \\ &= \frac{6}{2x+4} \end{aligned}$$

$$gf(x) = \frac{3}{x+2}$$

b)

$f(x)$ has to be a valid input for g

$$2x+3 > -1$$

$$2x > -4$$

$$x > -2$$

$x+2$ takes every positive value

$$gf(x) > 0$$

c)

$$\begin{aligned} fg(x) &= 2x \times \frac{6}{x+1} + 3 \\ &= \frac{12}{x+1} + 3 \end{aligned}$$

fg needs $x > -1$, gf needs $x > -2$: so $x > -1$

$$\frac{12}{x+1} + 3 = \frac{3}{x+2}$$

Multiply through by $(x+1)(x+2)$

$$12(x+2) + 3(x+1)(x+2) = 3(x+1)$$

$$12x + 24 + 3x^2 + 9x + 6 = 3x + 3$$

$$3x^2 + 18x + 27 = 0$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)^2 = 0$$

$$x = -3$$

-3 is not > -1 , so it is not in the domain

$fg(x) = gf(x)$ has no solutions

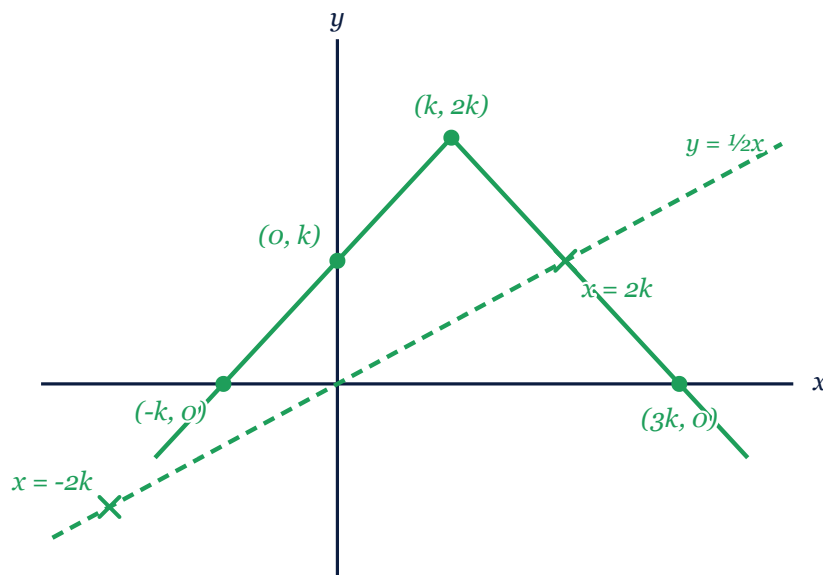
11

The function f is defined by

$$f(x) = 2k - |x - k|, \quad x \in \mathbb{R}$$

where k is a positive constant.

- Sketch the graph of $y = f(x)$, stating, in terms of k , the coordinates of the maximum point and of the points where the graph meets the coordinate axes.
- Solve, in terms of k , the equation $f(x) = \frac{1}{2}x$
- Hence find, in terms of k , the set of values of x for which $f(x) > \frac{1}{2}x$



(9 marks)

a)

Maximum where $|x - k| = 0$

maximum $(k, 2k)$

x-axis: $f(x) = 0$

$$|x - k| = 2k$$

$$x - k = 2k \text{ or } x - k = -2k$$

$$x = 3k \text{ or } x = -k$$

y-axis: $x = 0$, and $k > 0$

$$\begin{aligned} f(0) &= 2k - |-k| \\ &= k \end{aligned}$$

$(-k, 0), (3k, 0), (0, k)$

b)

Right branch, $x \geq k$

$$2k - (x - k) = \frac{1}{2}x$$

$$3k = \frac{3}{2}x$$

$$x = 2k$$

$2k \geq k$ so it is valid

Left branch, $x < k$

$$2k + (x - k) = \frac{1}{2}x$$

$$k = -\frac{1}{2}x$$

$$x = -2k$$

$-2k < k$ so it is valid

$$x = -2k \text{ or } x = 2k$$

c)

The graph is above the line between the two crossings

$$-2k < x < 2k$$

12

Mia makes a cup of coffee. Its temperature, $T^{\circ}\text{C}$, t minutes after it is made is modelled by

$$T = f(t) = 20 + 70e^{-0.04t}, t \geq 0$$

- State the range of f .
- Find $f^{-1}(T)$, stating its domain, and find the value of $f^{-1}(50)$, explaining what it means in this context.

A temperature of $T^{\circ}\text{C}$ is $F^{\circ}\text{F}$ where $F = g(T) = 1.8T + 32$

- Find $gf(t)$ in its simplest form, and hence find the time at which the coffee is at 140°F .

(9 marks)

a)

$$\begin{aligned} f(0) &= 20 + 70 \\ &= 90 \end{aligned}$$

$e^{-0.04t} \rightarrow 0$ as t gets large, but never reaches 0

$$20 < f(t) \leq 90$$

b)

$$\begin{aligned} T &= 20 + 70e^{-0.04t} \\ e^{-0.04t} &= \frac{T - 20}{70} \\ -0.04t &= \ln\left(\frac{T - 20}{70}\right) \\ t &= -25 \ln\left(\frac{T - 20}{70}\right) \\ &= 25 \ln\left(\frac{70}{T - 20}\right) \end{aligned}$$

$$f^{-1}(T) = 25 \ln\left(\frac{70}{T - 20}\right), 20 < T \leq 90$$

$$\begin{aligned} f^{-1}(50) &= 25 \ln\left(\frac{70}{30}\right) \\ &= 25 \ln\left(\frac{7}{3}\right) \\ &= 21.18... \end{aligned}$$

21.2 minutes: the time for the coffee to cool to 50°C

c)

$$\begin{aligned} gf(t) &= 1.8(20 + 70e^{-0.04t}) + 32 \\ &= 36 + 126e^{-0.04t} + 32 \end{aligned}$$

$$gf(t) = 68 + 126e^{-0.04t}$$

$$68 + 126e^{-0.04t} = 140$$

$$126e^{-0.04t} = 72$$

$$e^{-0.04t} = \frac{4}{7}$$

$$t = 25 \ln\left(\frac{7}{4}\right)$$

$$= 13.99\dots$$

$t = 14.0 \text{ minutes}$