



1 Find

a. $\int (6e^{3x} + \frac{5}{x}) dx, \quad x > 0$

b. $\int (3 \sin 2x + 4 \sec^2 x) dx$

c. $\int \sqrt{4x+1} dx$

(5 marks)

a) $\int e^{kx} dx = \frac{1}{k} e^{kx}$ and $\int \frac{1}{x} dx = \ln x$

$$\begin{aligned} \int (6e^{3x} + \frac{5}{x}) dx &= \frac{6}{3} e^{3x} + 5 \ln x + c \\ &= 2e^{3x} + 5 \ln x + c \end{aligned}$$

$$\boxed{2e^{3x} + 5 \ln x + c}$$

b) $\int \sin kx dx = -\frac{1}{k} \cos kx$ and $\int \sec^2 x dx = \tan x$

$$\int (3 \sin 2x + 4 \sec^2 x) dx = -\frac{3}{2} \cos 2x + 4 \tan x + c$$

$$\boxed{-\frac{3}{2} \cos 2x + 4 \tan x + c}$$

c) Write as a power, then reverse the chain rule

$$\begin{aligned} \int (4x+1)^{1/2} dx &= \frac{(4x+1)^{3/2}}{\frac{3}{2} \times 4} \\ &= \frac{(4x+1)^{3/2}}{6} + c \end{aligned}$$

$$\boxed{\frac{1}{6} (4x+1)^{3/2} + c}$$

2

Find the exact value of $\int_1^3 \frac{4x}{x^2+3} dx$, giving your answer in the form $\ln k$, where k is an integer.

(3 marks)

Numerator is 2 x the derivative of $x^2 + 3$, so use $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$

$$\int \frac{4x}{x^2+3} dx = 2 \ln(x^2+3)$$

Limits from 1 to 3

$$\begin{aligned} [2 \ln(x^2+3)] &= 2 \ln 12 - 2 \ln 4 \\ &= 2 \ln \frac{12}{4} \\ &= 2 \ln 3 \\ &= \ln 9 \end{aligned}$$

$$\boxed{\ln 9}$$

3

Using a suitable trigonometric identity, find the exact value of

$$\int_0^{\pi/3} 4 \sin^2 x dx$$

(4 marks)

Use $\cos 2x = 1 - 2 \sin^2 x$

$$2 \sin^2 x = 1 - \cos 2x$$

$$4 \sin^2 x = 2 - 2 \cos 2x$$

$$\int (2 - 2 \cos 2x) dx = 2x - \sin 2x$$

Limits from 0 to $\frac{\pi}{3}$

$$\begin{aligned} [2x - \sin 2x] &= \left(\frac{2\pi}{3} - \sin \frac{2\pi}{3} \right) - (0 - \sin 0) \\ &= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \end{aligned}$$

$$\boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

4

Find $\int x \cos 3x \, dx$

(4 marks)

Integration by parts: $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$

$$u = x$$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos 3x$$

$$v = \frac{1}{3} \sin 3x$$

$$\int x \cos 3x \, dx = \frac{x}{3} \sin 3x - \int \frac{1}{3} \sin 3x \, dx$$

$$= \frac{x}{3} \sin 3x - \frac{1}{3} \left(-\frac{1}{3} \cos 3x \right) + c$$

$$= \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + c$$

$\frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + c$

5

Given that k is a positive constant, show that

$$\int_k^{2k} \frac{x+k}{x^2} dx$$

is independent of k , and state its exact value.

(4 marks)

Split into separate terms

$$\frac{x+k}{x^2} = \frac{1}{x} + kx^{-2}$$

$$\int \left(\frac{1}{x} + kx^{-2} \right) dx = \ln x - \frac{k}{x}$$

Limits from k to $2k$

$$\left[\ln x - \frac{k}{x} \right] = \left(\ln 2k - \frac{k}{2k} \right) - \left(\ln k - \frac{k}{k} \right)$$

$$= \ln 2k - \ln k - \frac{1}{2} + 1$$

$$= \ln \frac{2k}{k} + \frac{1}{2}$$

$$= \ln 2 + \frac{1}{2}$$

No k left in the answer

$$\ln 2 + \frac{1}{2}, \text{ independent of } k$$

6

Find the exact value of

$$\int_1^e x^2 \ln x \, dx$$

giving your answer in the form $ae^3 + b$, where a and b are rational.

(5 marks)

Parts with $u = \ln x$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^2$$

$$v = \frac{x^3}{3}$$

$$\begin{aligned} \int x^2 \ln x \, dx &= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \times \frac{1}{x} \, dx \\ &= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx \\ &= \frac{x^3}{3} \ln x - \frac{x^3}{9} \end{aligned}$$

Limits from 1 to e , and $\ln e = 1$, $\ln 1 = 0$

$$\begin{aligned} \left[\frac{x^3}{3} \ln x - \frac{x^3}{9} \right] &= \left(\frac{e^3}{3} - \frac{e^3}{9} \right) - \left(0 - \frac{1}{9} \right) \\ &= \frac{2}{9}e^3 + \frac{1}{9} \end{aligned}$$

$$\boxed{\frac{2}{9}e^3 + \frac{1}{9}}$$

- a. Use the substitution $u^2 = 2x - 1$ to show that $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$ can be written as $\int_1^3 u^2 + 1 \frac{du}{2}$
- b. Hence find the exact value of $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$

(6 marks)

a) Differentiate the substitution

$$2u \frac{du}{dx} = 2$$

$$dx = u du$$

Rearrange for x

$$x = \frac{u^2 + 1}{2}$$

$$\sqrt{2x-1} = u$$

Change the limits

$$x = 1: u^2 = 1, u = 1$$

$$x = 5: u^2 = 9, u = 3$$

$$\begin{aligned} \int \frac{x}{\sqrt{2x-1}} dx &= \int \frac{\frac{u^2+1}{2}}{u} \times u du \\ &= \int \frac{u^2+1}{2} du \quad (u \text{ from } 1 \text{ to } 3) \end{aligned}$$

shown

b) Integrate

$$\int \frac{u^2+1}{2} du = \frac{1}{2} \left(\frac{u^3}{3} + u \right)$$

Limits from 1 to 3

$$\begin{aligned} \frac{1}{2} \left[\frac{u^3}{3} + u \right] &= \frac{1}{2} \left((9+3) - \left(\frac{1}{3} + 1 \right) \right) \\ &= \frac{1}{2} \times \frac{32}{3} \\ &= \frac{16}{3} \end{aligned}$$

$$\frac{16}{3}$$

8

$$f(x) = \frac{7x+5}{(x+2)(2x+1)}, \quad x \geq 0$$

a. Express $f(x)$ in partial fractions.

b. Hence show that $\int_0^4 f(x) dx$ equals $\ln k$, stating the value of the integer k .

(6 marks)

a) Write as two fractions and multiply through

$$\frac{7x+5}{(x+2)(2x+1)} \equiv \frac{A}{x+2} + \frac{B}{2x+1}$$

$$7x+5 \equiv A(2x+1) + B(x+2)$$

Put $x = -2$

$$-9 = -3A$$

$$A = 3$$

Put $x = -\frac{1}{2}$

$$\frac{3}{2} = \frac{3}{2}B$$

$$B = 1$$

$$\boxed{\frac{3}{x+2} + \frac{1}{2x+1}}$$

b) Integrate each term (divide by 2 for the $2x$)

$$\int \left(\frac{3}{x+2} + \frac{1}{2x+1} \right) dx = 3 \ln(x+2) + \frac{1}{2} \ln(2x+1)$$

Limits from 0 to 4

$$\left[3 \ln(x+2) + \frac{1}{2} \ln(2x+1) \right] = (3 \ln 6 + \frac{1}{2} \ln 9) - (3 \ln 2 + \frac{1}{2} \ln 1)$$

$$= 3 \ln \frac{6}{2} + \ln 3$$

$$= 3 \ln 3 + \ln 3$$

$$= 4 \ln 3$$

$$= \ln 81$$

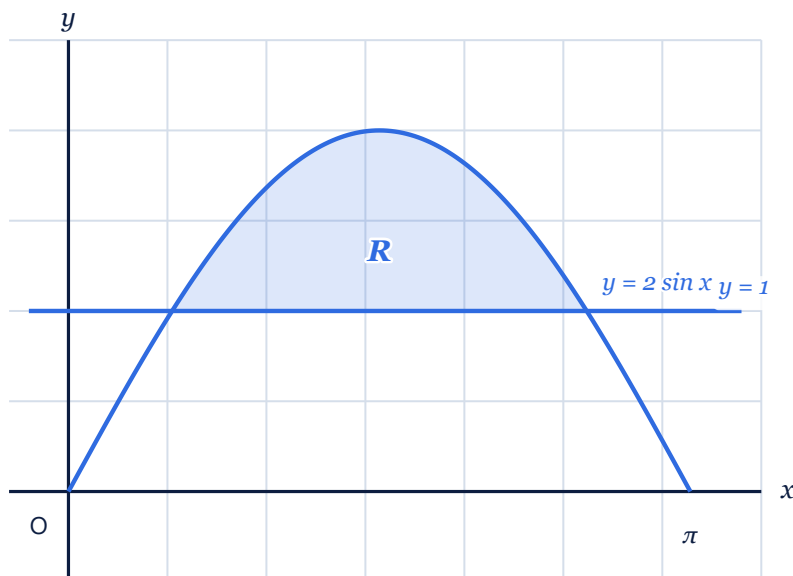
$$\boxed{k = 81}$$

9

The diagram shows part of the curve with equation $y = 2 \sin x$, $0 \leq x \leq \pi$, and the line with equation $y = 1$.

The region R , shown shaded, is bounded by the curve and the line.

- Find the exact x coordinates of the points where the line meets the curve.
- Find the exact area of R , giving your answer in the form $a\sqrt{3} - b\pi$, where a and b are rational.



(6 marks)

a) Set the curve equal to the line

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ or } x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\boxed{x = \frac{\pi}{6}, x = \frac{5\pi}{6}}$$

b) Area = $\int (\text{top} - \text{bottom}) dx$

$$\text{Area} = \int (2 \sin x - 1) dx$$

$$\int (2 \sin x - 1) dx = -2 \cos x - x$$

Limits from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$

$$\begin{aligned} [-2 \cos x - x] &= \left(-2 \cos \frac{5\pi}{6} - \frac{5\pi}{6}\right) - \left(-2 \cos \frac{\pi}{6} - \frac{\pi}{6}\right) \\ &= \left(\sqrt{3} - \frac{5\pi}{6}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) \\ &= 2\sqrt{3} - \frac{4\pi}{6} \end{aligned}$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$

$$\boxed{2\sqrt{3} - \frac{2\pi}{3}}$$

10

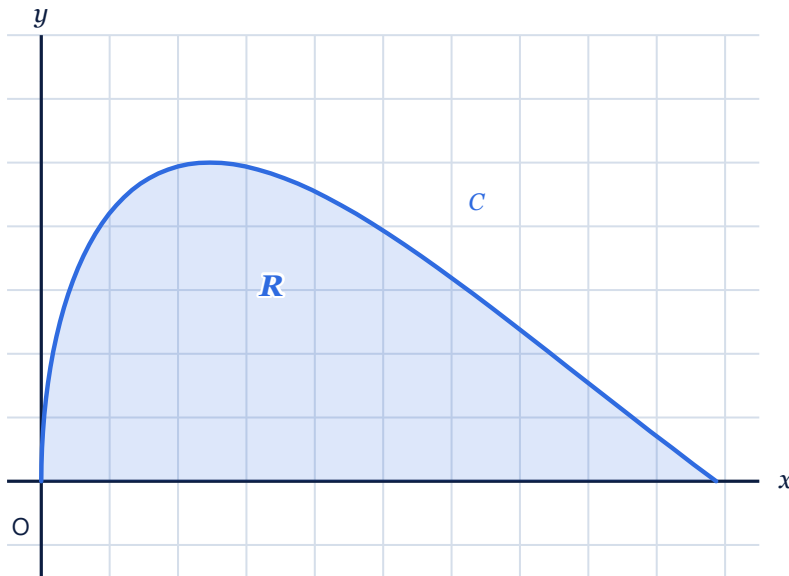
The curve C has parametric equations

$$x = t^2, \quad y = \sin t, \quad 0 \leq t \leq \pi$$

The region R , shown shaded in the diagram, is bounded by C and the x -axis.

a. Show that the area of R is given by $\int_0^\pi 2t \sin t \, dt$

b. Hence find the exact area of R .



(7 marks)

$$a) \text{ Area} = \int y \, dx = \int y \frac{dx}{dt} \, dt$$

$$\frac{dx}{dt} = 2t$$

$x = 0$ when $t = 0$ and $x = \pi^2$ when $t = \pi$, and $y \geq 0$ throughout

$$\begin{aligned} \text{Area} &= \int \sin t \times 2t \, dt \quad (t \text{ from } 0 \text{ to } \pi) \\ &= \int 2t \sin t \, dt \end{aligned}$$

shown

b) Integration by parts

$$u = 2t$$

$$\frac{du}{dt} = 2$$

$$\frac{dv}{dt} = \sin t$$

$$v = -\cos t$$

$$\begin{aligned} \int 2t \sin t \, dt &= -2t \cos t - \int -2 \cos t \, dt \\ &= -2t \cos t + 2 \sin t \end{aligned}$$

Limits from 0 to π

$$\begin{aligned} [-2t \cos t + 2 \sin t] &= (-2\pi \cos \pi + 2 \sin \pi) - (0 + 0) \\ &= 2\pi \end{aligned}$$

$$\boxed{2\pi}$$

11

The gates at a summer music festival open at 10 am. The rate at which people arrive, R hundred people per hour, t hours after the gates open is modelled by

$$R = 20te^{-0.5t}, \quad 0 \leq t \leq 4$$

- Using differentiation, find, according to the model, the greatest rate of arrival, giving your answer in hundreds of people per hour to 3 significant figures.
- Show that, according to the model, the number of people who arrive in the first 4 hours is $100(80 - 240e^{-2})$
- Hence find the number of people, to the nearest 10, who arrive in the first 4 hours.

(8 marks)

a) Product rule

$$\begin{aligned} \frac{dR}{dt} &= 20e^{-0.5t} + 20t(-0.5e^{-0.5t}) \\ &= 10e^{-0.5t}(2 - t) \end{aligned}$$

Set $\frac{dR}{dt} = 0$ ($e^{-0.5t}$ is never 0)

$$t = 2$$

$$R = 40e^{-1}$$

$$= 14.715\dots$$

$\frac{dR}{dt} > 0$ for $t < 2$ and < 0 for $t > 2$, so a maximum

14.7 hundred people per hour

b) Total (in hundreds) = $\int R \, dt$ from 0 to 4; integrate by parts

$$u = 20t$$

$$\frac{du}{dt} = 20$$

$$\frac{dv}{dt} = e^{-0.5t}$$

$$v = -2e^{-0.5t}$$

$$\begin{aligned} \int 20te^{-0.5t} \, dt &= -40te^{-0.5t} - \int -40e^{-0.5t} \, dt \\ &= -40te^{-0.5t} - 80e^{-0.5t} \end{aligned}$$

Limits from 0 to 4

$$\begin{aligned} [-40te^{-0.5t} - 80e^{-0.5t}] &= (-160e^{-2} - 80e^{-2}) - (0 - 80) \\ &= 80 - 240e^{-2} \text{ hundred people} \end{aligned}$$

Multiply by 100 for people

$100(80 - 240e^{-2})$ shown

c) Evaluate

$$100(80 - 240e^{-2}) = 4751.95...$$

4750 people

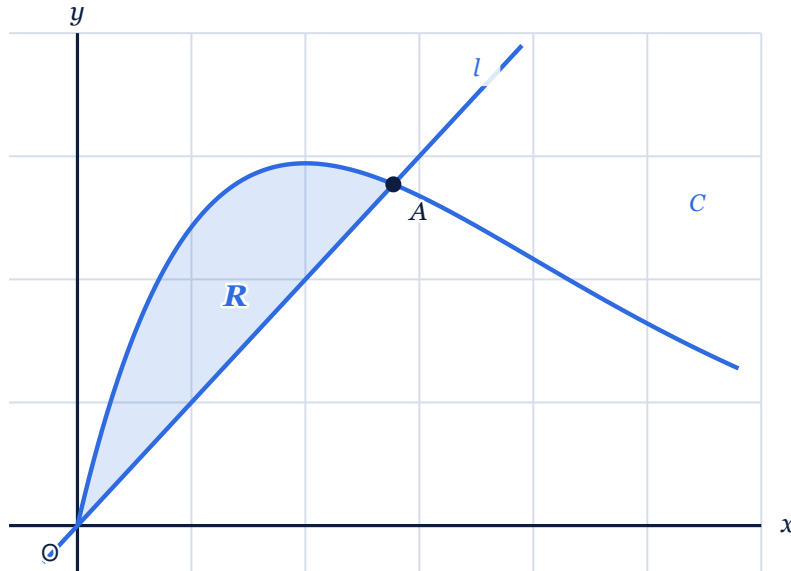
12

The diagram shows part of the curve C with equation $y = 4x e^{-x}$, $x \geq 0$, and the line l with equation $y = x$.

C and l meet at the origin O and at the point A .

The finite region R , shown shaded, is bounded by C and l .

- Find the exact x coordinate of A .
- Show that the exact area of R is $3 - 2 \ln 2 - 2(\ln 2)^2$



(9 marks)

a) Set the curve equal to the line

$$4x e^{-x} = x$$

$$x(4e^{-x} - 1) = 0$$

$x = 0$ is 0 , so

$$e^{-x} = \frac{1}{4}$$

$$-x = \ln \frac{1}{4}$$

$$x = \ln 4$$

$$x = \ln 4$$

b) Area = $\int (\text{curve} - \text{line}) dx$ from 0 to $\ln 4$

$$\text{Area} = \int (4x e^{-x} - x) dx$$

Integrate $4x e^{-x}$ by parts

$$u = 4x$$

$$\frac{du}{dx} = 4$$

$$\frac{dv}{dx} = e^{-x}$$

$$v = -e^{-x}$$

$$\begin{aligned}\int 4x e^{-x} dx &= -4x e^{-x} - \int -4e^{-x} dx \\ &= -4x e^{-x} - 4e^{-x}\end{aligned}$$

$$\text{At } x = \ln 4, e^{-x} = \frac{1}{4}$$

$$\begin{aligned}[-4x e^{-x} - 4e^{-x}] &= (-\ln 4 - 1) - (0 - 4) \\ &= 3 - \ln 4\end{aligned}$$

$$\int x dx = \left[\frac{x^2}{2} \right] = \frac{(\ln 4)^2}{2}$$

$$\ln 4 = 2 \ln 2$$

$$\frac{(\ln 4)^2}{2} = \frac{4(\ln 2)^2}{2} = 2(\ln 2)^2$$

$$\text{Area} = 3 - 2 \ln 2 - 2(\ln 2)^2$$

$$\boxed{3 - 2 \ln 2 - 2(\ln 2)^2 \text{ shown}}$$