



1 $f(x) = x^3 + 2x - 7$

- Show that the equation $f(x) = 0$ has a root α in the interval $[1.5, 1.6]$
- By choosing a suitable interval, show that $\alpha = 1.57$ correct to 2 decimal places.

(4 marks)

a)

Substitute both ends of the interval

$$f(1.5) = 1.5^3 + 2(1.5) - 7$$

$$f(1.5) = -0.625$$

$$f(1.6) = 1.6^3 + 2(1.6) - 7$$

$$f(1.6) = 0.296$$

Change of sign and f is continuous (a polynomial)

So there is a root α with $1.5 < \alpha < 1.6$

b)

Test the bounds for 1.57 to 2 dp: 1.565 and 1.575

$$f(1.565) = -0.0370$$

$$f(1.575) = 0.0570$$

Change of sign, f continuous

$$1.565 < \alpha < 1.575$$

$$\alpha = 1.57 \text{ (2 dp)}$$

2

The equation $x^3 - 6x + 2 = 0$ has a root α between 0 and 1

a. Show that the equation can be written in the form $x = \frac{x^3 + 2}{6}$

b. Use the iteration formula $x_{n+1} = \frac{x_n^3 + 2}{6}$ with $x_1 = 0.5$ to find the values of x_2 , x_3 and x_4 , giving each answer to 4 decimal places.

c. Write down the value of α correct to 3 decimal places.

(5 marks)

a)

$$x^3 - 6x + 2 = 0$$

$$x^3 + 2 = 6x$$

$$x = \frac{(x^3 + 2)}{6}$$

b)

Substitute each answer back in (keep the full value on the calculator)

$$x_2 = \frac{(0.5^3 + 2)}{6}$$

$$x_2 = 0.354166...$$

$$x_2 = 0.3542$$

$$x_3 = \frac{(0.354166...^3 + 2)}{6}$$

$$x_3 = 0.3407$$

$$x_4 = 0.3399$$

c)

Keep iterating: $x_5 = 0.33988...$, $x_6 = 0.33988...$

$$\alpha = 0.340 \text{ (3 dp)}$$

3

The equation $x^4 - 3x - 5 = 0$ has a root β close to $x = 2$

Taking $x_1 = 2$ as a first approximation, apply the Newton-Raphson method twice to find x_2 and x_3 .

Give each answer to 4 decimal places.

(4 marks)

Differentiate

$$f(x) = x^4 - 3x - 5$$

$$f'(x) = 4x^3 - 3$$

$$\text{Newton-Raphson: } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(2) = 16 - 6 - 5 = 5$$

$$f'(2) = 32 - 3 = 29$$

$$x_2 = 2 - \frac{5}{29}$$

$$x_2 = 1.827586...$$

Repeat with x_2 (full calculator value)

$$f(1.827586...) = 0.67332...$$

$$f'(1.827586...) = 21.4171...$$

$$x_3 = 1.827586... - \frac{0.67332...}{21.4171...}$$

$$x_3 = 1.79614...$$

$$x_2 = 1.8276, x_3 = 1.7961$$

4

The table shows values of $y = e^{-x^2}$ for $0 \leq x \leq 1$, with values of y given to 4 decimal places.

x	0	0.25	0.5	0.75	1
y	1	0.9394		0.5698	0.3679

a. Complete the table, giving the missing value of y to 4 decimal places.

b. Use the trapezium rule with all the values of y in the completed table to find an estimate for $\int_0^1 e^{-x^2} dx$, giving your answer to 3 significant figures.

(4 marks)

a)

$$y = e^{-0.5^2} = e^{-0.25}$$

$$y = 0.7788$$

b)

Strip width $h = 0.25$

Trapezium rule: $\frac{h}{2} [\text{first} + \text{last} + 2(\text{middle ones})]$

$$\text{Area} \approx \frac{1}{2} \times 0.25 \times [1 + 0.3679 + 2(0.9394 + 0.7788 + 0.5698)]$$

$$\text{Area} \approx 0.125 \times [1.3679 + 2(2.288)]$$

$$\text{Area} \approx 0.125 \times [1.3679 + 4.576]$$

$$\text{Area} \approx 0.125 \times 5.9439$$

$$\text{Area} \approx 0.74299$$

$$\boxed{0.743 \text{ (3 s.f.)}}$$

5

The curve C has equation $y = e^{0.5x} - x^2$

C has a turning point P whose x coordinate α lies between 0 and 1

- Show that α is a solution of the equation $x = 0.25e^{0.5x}$
- Using the iteration formula $x_{n+1} = 0.25e^{0.5x_n}$ with $x_1 = 0.3$, find the values of x_2 , x_3 and x_4 , giving each answer to 4 decimal places.
- Using a suitable interval and a suitable function that should be stated, show that $\alpha = 0.2888$ to 4 decimal places.

(8 marks)

a)

At a turning point $dy/dx = 0$

$$\frac{dy}{dx} = 0.5e^{0.5x} - 2x$$

$$0.5e^{0.5x} - 2x = 0$$

$$2x = 0.5e^{0.5x}$$

$$x = 0.25e^{0.5x}$$

b)

$$x_2 = 0.25e^{0.5 \times 0.3}$$

$$x_2 = 0.29046... = 0.2905$$

$$x_3 = 0.25e^{0.5 \times 0.29046...}$$

$$x_3 = 0.28908... = 0.2891$$

$$x_4 = 0.28888... = 0.2889$$

c)

Use the gradient function $g(x) = 0.5e^{0.5x} - 2x$

Bounds for 0.2888 to 4 dp: 0.28875 and 0.28885

$$g(0.28875) = 0.000159 > 0$$

$$g(0.28885) = -0.0000125 < 0$$

Change of sign and g is continuous

$$0.28875 < \alpha < 0.28885$$

$$\alpha = 0.2888 \text{ (4 dp)}$$

6

(i) $f(x) = \frac{3}{x-2} + x, \quad x \neq 2$

A student finds $f(1.9)$ and $f(2.1)$ and says that, because the signs are different, the equation $f(x) = 0$ has a root between 1.9 and 2.1

a. Find $f(1.9)$ and $f(2.1)$

b. Explain why the student's conclusion is not correct.

(ii) $g(x) = 4x^2 - 16x + 15$

c. Show that $g(1)$ and $g(3)$ have the same sign.

d. By considering $g(2)$, explain why the equation $g(x) = 0$ has at least two roots between 1 and 3, even though $g(1)$ and $g(3)$ have the same sign.

(6 marks)

a)

$$f(1.9) = \frac{3}{(1.9-2)} + 1.9$$

$$f(1.9) = -30 + 1.9 = -28.1$$

$$f(2.1) = \frac{3}{(2.1-2)} + 2.1$$

$$f(2.1) = 30 + 2.1 = 32.1$$

b)

$x = 2$ makes the denominator zero

There is a vertical asymptote at $x = 2$

So f is not continuous on $[1.9, 2.1]$

The sign changes because of the asymptote, not a root

c)

$$g(1) = 4 - 16 + 15 = 3$$

$$g(3) = 36 - 48 + 15 = 3$$

Both positive: same sign

d)

$$g(2) = 16 - 32 + 15 = -1$$

$g(1) > 0, g(2) < 0$: sign change on $[1, 2]$

$g(2) < 0, g(3) > 0$: sign change on $[2, 3]$

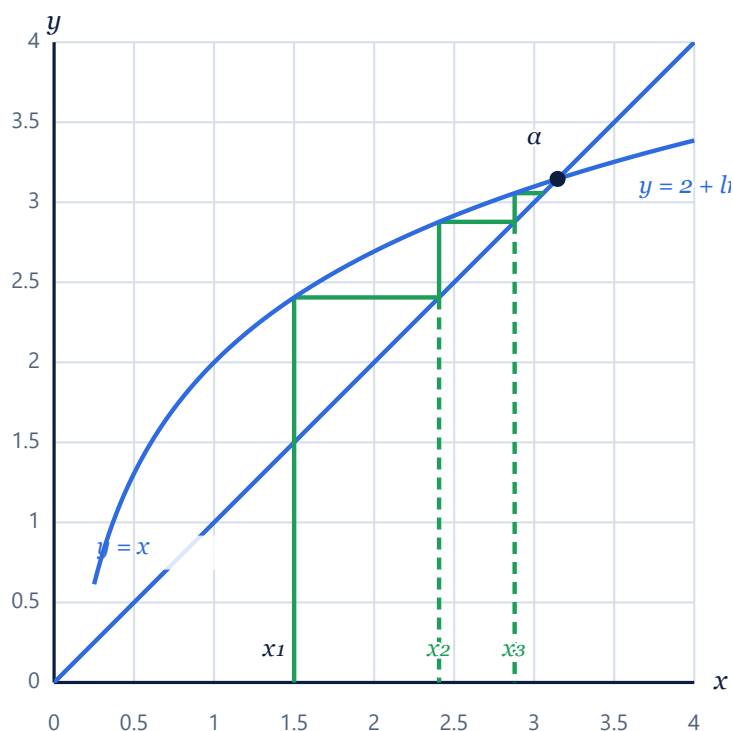
g is a polynomial, so continuous

At least one root in each interval: at least two roots

7

The diagram shows part of the curve with equation $y = 2 + \ln x$ and the line $y = x$. They meet at the point where $x = \alpha$

- Show that α lies between 3 and 3.5
- Use the iteration formula $x_{n+1} = 2 + \ln x_n$ with $x_1 = 1.5$ to find x_2 and x_3 , giving each answer to 4 decimal places.
- On the diagram, draw a staircase diagram to show how the iteration in part (b) moves towards α . Mark x_2 and x_3 on the x -axis.
- A different rearrangement gives the iteration formula $x_{n+1} = e^{x_n - 2}$. Using $x_1 = 3$, find x_2 , x_3 and x_4 to 3 decimal places, and explain why this formula cannot be used to find α .



(9 marks)

a)

Let $f(x) = x - 2 - \ln x$ (roots of f are where the graphs meet)

$$f(3) = 3 - 2 - \ln 3 = -0.0986$$

$$f(3.5) = 3.5 - 2 - \ln 3.5 = 0.2472$$

Change of sign and f is continuous for $x > 0$

$$3 < \alpha < 3.5$$

b)

$$x_2 = 2 + \ln 1.5 = 2.4055$$

$$x_3 = 2 + \ln 2.4055... = 2.8777$$

c)

Up from $x_1 = 1.5$ to the curve

Across to the line $y = x$, then down to the x -axis: x_2

Up to the curve again, across to $y = x$: x_3

A staircase climbing towards α

d)

$$x_2 = e^{3-2} = 2.718$$

$$x_3 = e^{2.718...-2} = 2.051$$

$$x_4 = e^{2.051...-2} = 1.052$$

α is about 3.15

The values move further from α each time: this iteration diverges from α

8

$$f(x) = x^3 - 3x^2 + 5$$

The equation $f(x) = 0$ has exactly one real root γ

a. Show that, for this equation, the Newton-Raphson formula can be written

$$x_{n+1} = \frac{2x_n^3 - 3x_n^2 - 5}{3x_n^2 - 6x_n}$$

b. Using this formula with $x_1 = -1$, find x_2 and x_3 , giving each answer to 4 decimal places.

c. Explain why the Newton-Raphson method cannot be used with $x_1 = 2$

(6 marks)

a)

$$f'(x) = 3x^2 - 6x$$

Newton-Raphson

$$x_{n+1} = x_n - \frac{(x_n^3 - 3x_n^2 + 5)}{(3x_n^2 - 6x_n)}$$

Common denominator

$$x_{n+1} = \frac{(x_n(3x_n^2 - 6x_n) - (x_n^3 - 3x_n^2 + 5))}{(3x_n^2 - 6x_n)}$$

$$x_{n+1} = \frac{(3x_n^3 - 6x_n^2 - x_n^3 + 3x_n^2 - 5)}{(3x_n^2 - 6x_n)}$$

$$x_{n+1} = \frac{(2x_n^3 - 3x_n^2 - 5)}{(3x_n^2 - 6x_n)}$$

b)

$$x_2 = \frac{(2(-1)^3 - 3(-1)^2 - 5)}{(3(-1)^2 - 6(-1))}$$

$$x_2 = \frac{(-2 - 3 - 5)}{(3 + 6)}$$

$$x_2 = -\frac{10}{9} = -1.1111$$

$$x_3 = -1.10383...$$

$$x_2 = -1.1111, x_3 = -1.1038$$

c)

$$f'(2) = 3(4) - 6(2) = 0$$

The denominator is zero, so x_2 cannot be found

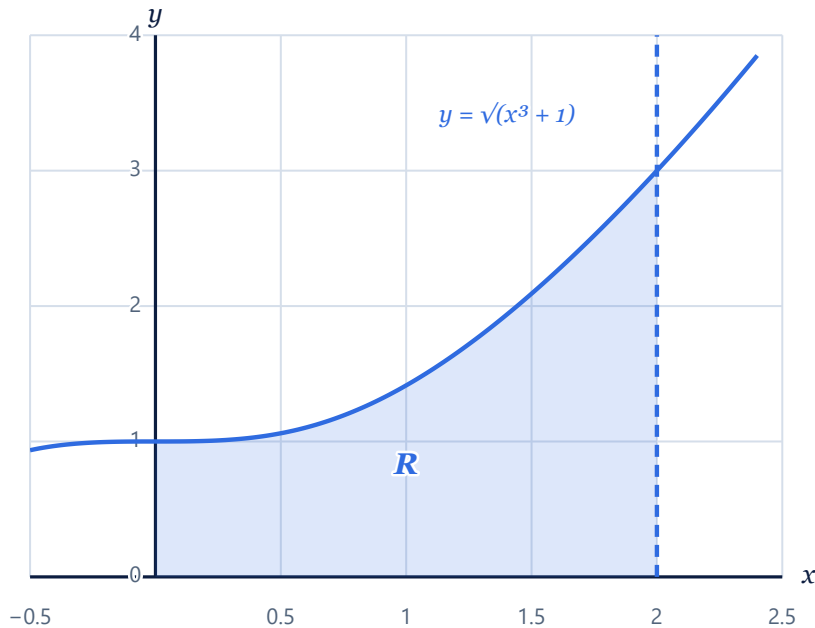
The tangent at $x = 2$ is horizontal, so it never meets the x -axis

9

The diagram shows part of the curve with equation $y = \sqrt{x^3 + 1}$. The region R , shown shaded, is bounded by the curve, the x -axis, the y -axis and the line $x = 2$

x	0	0.5	1	1.5	2
y	1		1.4142		3

- Complete the table, giving the values of y to 4 decimal places.
- Use the trapezium rule with all the values in the table to find an estimate for the area of R , giving your answer to 3 significant figures.
- Using the diagram, state with a reason whether your answer to part (b) is an overestimate or an underestimate.
- Explain how the trapezium rule could be used to give a more accurate estimate.



(7 marks)

a)

$$x = 0.5: y = \sqrt{(0.5^3 + 1)} = \sqrt{1.125} = 1.0607$$

$$x = 1.5: y = \sqrt{(1.5^3 + 1)} = \sqrt{4.375} = 2.0917$$

b)

$$h = 0.5$$

$$\text{Area} \approx \frac{0.5}{2} [1 + 3 + 2(1.0607 + 1.4142 + 2.0917)]$$

$$\text{Area} \approx 0.25 [4 + 2(4.5666)]$$

$$\text{Area} \approx 0.25 \times 13.1332$$

$$\text{Area} \approx 3.2833$$

$$\boxed{\text{Area} \approx 3.28 \text{ (3 s.f.)}}$$

c)

The curve bends upwards (convex) between $x = 0$ and $x = 2$

The straight top of each trapezium lies above the curve

Overestimate

d)

Use more strips, each of smaller width

10

The table shows some values of x and y for a continuous curve with equation $y = f(x)$

x	1	1.5	2	2.5	3
y	2.4	3.1	3.5	3.2	2.6

a. Use the trapezium rule with all the values in the table to find an estimate for $\int_1^3 f(x) dx$

b. Using your answer to part (a), find an estimate for $\int_1^3 (3f(x) - 2x) dx$

c. Using your answer to part (a), write down an estimate for $\int_0^2 f(x+1) dx$, giving a reason.

(6 marks)

a)

$$h = 0.5$$

$$\int f(x) dx \approx \frac{0.5}{2} [2.4 + 2.6 + 2(3.1 + 3.5 + 3.2)]$$

$$\int f(x) dx \approx 0.25 [5 + 19.6]$$

$$\int f(x) dx \approx 0.25 \times 24.6$$

6.15

b)

Split the integral

$$\int (3f(x) - 2x) dx = 3 \int f(x) dx - \int 2x dx$$

$$\int 2x dx = [x^2] \text{ from 1 to 3}$$

$$\int 2x dx = 9 - 1 = 8$$

$$\int (3f(x) - 2x) dx \approx 3(6.15) - 8$$

$$\int (3f(x) - 2x) dx \approx 18.45 - 8$$

10.45

c)

$y = f(x+1)$ is $y = f(x)$ translated 1 unit left

$x = 0$ to 2 on the new curve matches $x = 1$ to 3 on the old one

6.15

11

The region R is bounded by the curve with equation $y = x \ln x$, the x -axis and the line $x = 3$

- Show that the curve meets the x -axis at $x = 1$ for $x > 0$
- Use the trapezium rule with four strips of equal width to find an estimate for the area of R , giving your answer to 4 decimal places.
- Use integration by parts to find the exact area of R , giving your answer in the form $a \ln 3 + b$, where a and b are rational.
- Find the percentage error in the estimate from part (b), and explain, by considering $\frac{d^2y}{dx^2}$, why the trapezium rule gives an overestimate here.

(10 marks)

a)

$$x \ln x = 0$$

$x > 0$, so $x \neq 0$

$$\ln x = 0$$

$$x = 1$$

b)

From $x = 1$ to 3 with 4 strips: $h = 0.5$

x : 1, 1.5, 2, 2.5, 3

y : 0, 0.6082, 1.3863, 2.2907, 3.2958

$$\text{Area} \approx \frac{0.5}{2} [0 + 3.2958 + 2(0.6082 + 1.3863 + 2.2907)]$$

$$\text{Area} \approx 0.25 [3.2958 + 8.5704]$$

$$\text{Area} \approx 0.25 \times 11.8662$$

$$\boxed{\text{Area} \approx 2.9666}$$

c)

Integration by parts: $u = \ln x$, $\frac{dv}{dx} = \frac{1}{x}$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{1}{2} x^2$$

$$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \times \frac{1}{x} \, dx$$

$$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x \, dx$$

$$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$$

Limits: $\left[\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right]$ from 1 to 3

$$\text{Area} = \left(\frac{9}{2} \ln 3 - \frac{9}{4} \right) - \left(\frac{1}{2} \ln 1 - \frac{1}{4} \right)$$

$$\text{Area} = \frac{9}{2} \ln 3 - \frac{9}{4} + \frac{1}{4}$$

$$\boxed{\text{Area} = \frac{9}{2} \ln 3 - 2}$$

d)

$$\frac{9}{2} \ln 3 - 2 = 2.94376...$$

$$\% \text{ error} = \frac{(2.9666 - 2.94376)}{2.94376} \times 100$$

$$\% \text{ error} = 0.776\%$$

Second derivative

$$\frac{dy}{dx} = \ln x + 1$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

$1/x > 0$ for $x > 0$, so the curve is convex (bends upwards)

$\boxed{\text{The trapezium tops lie above the curve: overestimate}}$

12

The concentration of a medicine in a patient's blood, C mg per litre, t hours after it is taken, is modelled by

$$C = 8te^{-0.5t}, \quad t \geq 0$$

- a. Find the maximum concentration according to the model, giving your answer to 3 significant figures.

The medicine is effective while $C \geq 4$. The concentration first reaches 4 mg per litre when $t = 0.715$ (3 decimal places). It falls back to 4 mg per litre when $t = T$

- b. Show that T is a root of the equation $h(t) = 0$, where $h(t) = 2te^{-0.5t} - 1$, and show that $4 < T < 5$
 c. Taking $t_1 = 4.5$ as a first approximation, apply the Newton-Raphson method twice to find an approximation for T , giving your answer to 3 decimal places.
 d. Hence find, giving your answer in hours to 2 decimal places, how long the medicine is effective for.

(10 marks)

a)

Product rule

$$\frac{dC}{dt} = 8e^{-0.5t} + 8t(-0.5e^{-0.5t})$$

$$\frac{dC}{dt} = 8e^{-0.5t} - 4te^{-0.5t}$$

$$\frac{dC}{dt} = 4e^{-0.5t}(2 - t)$$

$e^{-0.5t} > 0$, so $dC/dt = 0$ when $t = 2$

$$C = 8(2)e^{-1}$$

$$C = 5.886...$$

$$\boxed{5.89 \text{ mg per Litre}}$$

b)

$$8te^{-0.5t} = 4$$

$$2te^{-0.5t} = 1$$

$$2te^{-0.5t} - 1 = 0$$

$$h(4) = 8e^{-2} - 1 = 0.0827$$

$$h(5) = 10e^{-2.5} - 1 = -0.179$$

Change of sign, h continuous

$$\boxed{4 < T < 5}$$

c)

Differentiate h (product rule)

$$h'(t) = 2e^{-0.5t} - te^{-0.5t}$$

$$h(4.5) = -0.051407...$$

$$h'(4.5) = -0.263498...$$

$$t_2 = 4.5 - \frac{-0.051407...}{-0.263498...}$$

$$t_2 = 4.30491...$$

$$t_3 = 4.30658...$$

$$T = 4.307 \text{ (3 dp)}$$

d)

$$\text{Time} = 4.307 - 0.715$$

$$\text{Time} = 3.592 \text{ hours}$$

$$3.59 \text{ hours (2 dp)}$$