



- 1 A curve C has parametric equations

$$x = 2t - 1, \quad y = t^2 + 3, \quad t \in \mathbb{R}$$

- Find the coordinates of the point on C where $t = 3$
- Find a Cartesian equation for C in the form $y = f(x)$

(3 marks)

a)

$$x = 2(3) - 1 = 5$$

$$y = 3^2 + 3 = 12$$

$$(5, 12)$$

b)

Make t the subject of the simpler equation

$$x = 2t - 1$$

$$t = \frac{x + 1}{2}$$

Substitute into y

$$y = \left(\frac{x + 1}{2}\right)^2 + 3$$

$$y = \frac{(x + 1)^2}{4} + 3$$

2

A curve C has parametric equations

$$x = 1 + 4 \cos \theta, \quad y = 4 \sin \theta - 2, \quad 0 \leq \theta < 2\pi$$

Show that C is a circle and state its centre and radius.

(3 marks)

Make $\cos \theta$ and $\sin \theta$ the subjects

$$\cos \theta = \frac{x - 1}{4}$$

$$\sin \theta = \frac{y + 2}{4}$$

Use $\sin^2 \theta + \cos^2 \theta = 1$

$$\left(\frac{x - 1}{4}\right)^2 + \left(\frac{y + 2}{4}\right)^2 = 1$$

$$(x - 1)^2 + (y + 2)^2 = 16$$

circle, centre $(1, -2)$, radius 4

3

A curve C has parametric equations

$$x = t^3 - 2t, \quad y = t^2 + 4t$$

a. Find $\frac{dy}{dx}$ in terms of t b. Find an equation of the tangent to C at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$, where a , b and c are integers.

(5 marks)

a)

$$\frac{dx}{dt} = 3t^2 - 2$$

$$\frac{dy}{dt} = 2t + 4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\boxed{\frac{dy}{dx} = \frac{2t + 4}{3t^2 - 2}}$$

b)

Point where $t = 2$

$$x = 8 - 4 = 4$$

$$y = 4 + 8 = 12$$

Gradient where $t = 2$

$$\begin{aligned} \frac{dy}{dx} &= \frac{4 + 4}{12 - 2} \\ &= \frac{8}{10} = \frac{4}{5} \end{aligned}$$

$$y - y_1 = m(x - x_1)$$

$$y - 12 = \frac{4}{5}(x - 4)$$

$$5y - 60 = 4x - 16$$

$$\boxed{4x - 5y + 44 = 0}$$

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A curve C has parametric equations

$$x = 1 + 2 \sin t, \quad y = 3 - \cos 2t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

- Show that a Cartesian equation of C is $y = 2 + \frac{1}{2}(x - 1)^2$
- State the domain of values of x and the range of values of y for C .
- Explain why C does not include every point of the curve with equation $y = 2 + \frac{1}{2}(x - 1)^2, x \in \mathbb{R}$

(5 marks)

a)

$$\cos 2t = 1 - 2 \sin^2 t$$

$$\begin{aligned} y &= 3 - (1 - 2 \sin^2 t) \\ &= 2 + 2 \sin^2 t \end{aligned}$$

$$\text{From } x: \sin t = \frac{x - 1}{2}$$

$$\begin{aligned} y &= 2 + 2\left(\frac{x - 1}{2}\right)^2 \\ &= 2 + \frac{2(x - 1)^2}{4} \\ &= 2 + \frac{1}{2}(x - 1)^2 \end{aligned}$$

shown

b)

 $-1 \leq \sin t \leq 1$ on this interval

$$x = 1 + 2(-1) = -1$$

$$x = 1 + 2(1) = 3$$

$$0 \leq \sin^2 t \leq 1$$

$$y = 2 + 0 = 2$$

$$y = 2 + 2 = 4$$

$$-1 \leq x \leq 3, \quad 2 \leq y \leq 4$$

c)

 $x = 1 + 2 \sin t$ can never be less than -1 or more than 3
 C is only the part of the parabola with $-1 \leq x \leq 3$

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A curve C has parametric equations

$$x = \frac{3}{t-1}, \quad y = \frac{t^2}{t-1}, \quad t > 1$$

Show that a Cartesian equation of C is

$$y = \frac{(x+a)^2}{bx}, \quad x > 0$$

where a and b are integers to be found.

(3 marks)

Make t the subject

$$t - 1 = \frac{3}{x}$$

$$t = \frac{3}{x} + 1$$

$$= \frac{x+3}{x}$$

Substitute into y

$$y = t^2 \div (t-1)$$

$$= \frac{(x+3)^2}{x^2} \div \frac{3}{x}$$

$$= \frac{(x+3)^2}{x^2} \times \frac{x}{3}$$

$$= \frac{(x+3)^2}{3x}$$

$$a = 3, b = 3$$

6

A curve C has parametric equations

$$x = t^2 - 4t, \quad y = t^3 - 3t, \quad t \in \mathbb{R}$$

- Find the coordinates of the points on C where the tangent is parallel to the x -axis.
- Find the coordinates of the point on C where the tangent is parallel to the y -axis.

(6 marks)

$$\frac{dx}{dt} = 2t - 4$$

$$\frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 4}$$

a)

Parallel to the x -axis: $\frac{dy}{dx} = 0$, so the numerator is 0

$$3t^2 - 3 = 0$$

$$t = \pm 1$$

$$t = 1$$

$$x = 1 - 4 = -3$$

$$y = 1 - 3 = -2$$

$$t = -1$$

$$x = 1 + 4 = 5$$

$$y = -1 + 3 = 2$$

$$\boxed{(-3, -2) \text{ and } (5, 2)}$$

b)

Parallel to the y -axis: the denominator is 0

$$2t - 4 = 0$$

$$t = 2$$

$$x = 4 - 8 = -4$$

$$y = 8 - 6 = 2$$

$$\boxed{(-4, 2)}$$

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A curve C has parametric equations

$$x = t^2 + 1, \quad y = 2t - 1, \quad t \in \mathbb{R}$$

The point P on C has parameter $t = 2$

a. Show that the normal to C at P has equation $y = 13 - 2x$

b. The normal meets C again at the point Q . Find the coordinates of Q .

(7 marks)

a)

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$$

At $t = 2$

$$x = 4 + 1 = 5$$

$$y = 4 - 1 = 3$$

$$\frac{dy}{dx} = \frac{1}{2}$$

Normal gradient = $-1 \div$ tangent gradient

$$m = -2$$

$$y - 3 = -2(x - 5)$$

$$y = 13 - 2x$$

shown

b)

Substitute the parametric equations into the normal

$$2t - 1 = 13 - 2(t^2 + 1)$$

$$2t - 1 = 11 - 2t^2$$

$$2t^2 + 2t - 12 = 0$$

$$t^2 + t - 6 = 0$$

$$(t + 3)(t - 2) = 0$$

$t = 2$ is the point P

$$t = -3$$

$$x = 9 + 1 = 10$$

$$y = -6 - 1 = -7$$

Q(10, -7)

8

In this question you must show all stages of your working.

A curve C has parametric equations

$$x = 2 \cos t, \quad y = \sin 2t, \quad 0 \leq t < 2\pi$$

Find the exact coordinates of all the points where C meets the line with equation $y = \frac{\sqrt{3}}{2}x$

(5 marks)

Substitute the parametric equations into the line

$$\sin 2t = \frac{\sqrt{3}}{2} \times 2 \cos t$$

$$2 \sin t \cos t = \sqrt{3} \cos t$$

Factorise, do not divide by $\cos t$

$$\cos t (2 \sin t - \sqrt{3}) = 0$$

$$\cos t = 0$$

$$t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = 0, y = \sin \pi = 0$$

$$\sin t = \frac{\sqrt{3}}{2}$$

$$t = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$t = \frac{\pi}{3}$$

$$x = 2 \cos \frac{\pi}{3} = 1$$

$$y = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$t = \frac{2\pi}{3}$$

$$x = 2 \cos \frac{2\pi}{3} = -1$$

$$y = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$(0, 0), (1, \frac{\sqrt{3}}{2}), (-1, -\frac{\sqrt{3}}{2})$$

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A curve C has parametric equations

$$x = 2 \cos t, \quad y = 1 + \cos 2t, \quad 0 \leq t \leq \pi$$

- a. Show that all points on C satisfy $y = \frac{1}{2}x^2$ and state the domain of values of x for C .
- b. The line with equation $y = x + k$, where k is a constant, meets C at two distinct points. Find the range of possible values of k , writing your answer in set notation.

(7 marks)

a)

$$\cos 2t = 2 \cos^2 t - 1$$

$$\begin{aligned} y &= 1 + 2 \cos^2 t - 1 \\ &= 2 \cos^2 t \end{aligned}$$

$$\cos t = \frac{x}{2}$$

$$\begin{aligned} y &= 2 \left(\frac{x}{2} \right)^2 \\ &= \frac{1}{2} x^2 \end{aligned}$$

$$-1 \leq \cos t \leq 1 \text{ for } 0 \leq t \leq \pi$$

$$y = \frac{1}{2} x^2, \quad -2 \leq x \leq 2$$

b)

$$\frac{1}{2} x^2 = x + k$$

$$x^2 - 2x - 2k = 0$$

Two distinct roots: $b^2 - 4ac > 0$

$$4 + 8k > 0$$

$$k > -\frac{1}{2}$$

Both roots must lie in $-2 \leq x \leq 2$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 + 8k}}{2} \\ &= 1 \pm \sqrt{1 + 2k} \end{aligned}$$

Larger root at most 2

$$1 + \sqrt{1 + 2k} \leq 2$$

$$\sqrt{1 + 2k} \leq 1$$

$$k \leq 0$$

Then the smaller root $1 - \sqrt{1 + 2k} \geq 0$, so it is inside the domain

$$\{k : -\frac{1}{2} < k \leq 0\}$$

10 A curve C has parametric equations

$$x = \cos 2\theta, \quad y = \sec \theta, \quad 0 < \theta < \frac{\pi}{2}$$

a. Show that $\frac{dy}{dx} = k \sec^3 \theta$, where k is a constant to be found.

b. Find an equation of the tangent to C at the point where $y = 2$, giving your answer in the form $y = mx + c$

(6 marks)

a.)

$$\frac{dx}{d\theta} = -2 \sin 2\theta$$

$$= -4 \sin \theta \cos \theta$$

$$\frac{dy}{d\theta} = \sec \theta \tan \theta$$

$$= \frac{\sin \theta}{\cos^2 \theta}$$

$$\frac{dy}{dx} = \frac{\sin \theta}{\cos^2 \theta} \div (-4 \sin \theta \cos \theta)$$

$$= -\frac{1}{4 \cos^3 \theta}$$

$$= -\frac{1}{4} \sec^3 \theta$$

$$\boxed{k = -\frac{1}{4}}$$

b.)

$$\sec \theta = 2$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

$$x = \cos \frac{2\pi}{3} = -\frac{1}{2}$$

$$\frac{dy}{dx} = -\frac{1}{4} \times 2^3 = -2$$

$$y - 2 = -2\left(x + \frac{1}{2}\right)$$

$$y - 2 = -2x - 1$$

$$\boxed{y = 1 - 2x}$$

11

A reflector is fixed to the rim of a bicycle wheel of radius 0.3 m. As the bicycle moves along level ground, the position of the reflector is modelled by

$$x = 0.3(\theta - \sin \theta), \quad y = 0.3(1 - \cos \theta), \quad 0 < \theta < 2\pi$$

where x metres is the horizontal distance moved by the reflector, y metres is its height above the ground and θ is the angle in radians that the wheel has turned through.

- Show that $\frac{dy}{dx} = \cot\left(\frac{\theta}{2}\right)$
- Find the exact coordinates of the reflector at the moment its path has gradient 1
- The reflector is 0.45 m above the ground at two points in this interval. Find the gradient of the path at each of these points, giving exact answers.
- State one assumption made in this model.

(9 marks)

a)

$$\frac{dx}{d\theta} = 0.3(1 - \cos \theta)$$

$$\frac{dy}{d\theta} = 0.3 \sin \theta$$

$$\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta}$$

Use the double angle formulae with $\theta = 2 \times \frac{\theta}{2}$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$\frac{dy}{dx} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= \cot \frac{\theta}{2}$$

shown

b)

$$\cot \frac{\theta}{2} = 1$$

$$\tan \frac{\theta}{2} = 1$$

$$0 < \frac{\theta}{2} < \pi$$

$$\frac{\theta}{2} = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{2}$$

$$x = 0.3\left(\frac{\pi}{2} - \sin \frac{\pi}{2}\right) = 0.3\left(\frac{\pi}{2} - 1\right)$$

$$y = 0.3\left(1 - \cos \frac{\pi}{2}\right) = 0.3$$

$$\left(0.3\left(\frac{\pi}{2} - 1\right), 0.3\right)$$

c)

$$0.3(1 - \cos \theta) = 0.45$$

$$1 - \cos \theta = 1.5$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

$$\frac{dy}{dx} = \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{4\pi}{3}$$

$$\frac{dy}{dx} = \cot \frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} \text{ and } -\frac{1}{\sqrt{3}}$$

d)

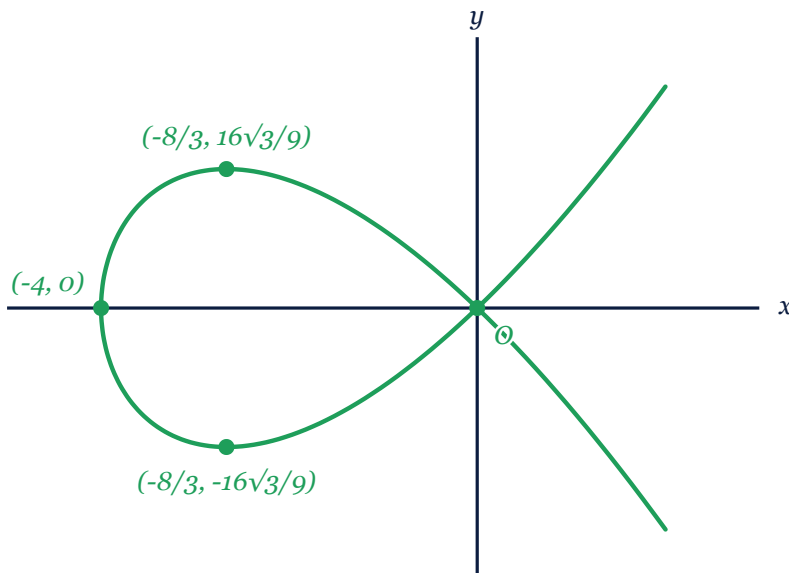
the wheel rolls without slipping on level ground

12

A curve C has parametric equations

$$x = t^2 - 4, \quad y = t^3 - 4t, \quad t \in \mathbb{R}$$

- Find the values of t at which C meets the x -axis, and hence show that C passes through the origin O twice.
- Find the gradients of the two tangents to C at O .
- Show that a Cartesian equation of C is $y^2 = x^2(x + 4)$.
- Find the exact coordinates of the points on C where the tangent is parallel to the x -axis.
- Sketch C for $-2.5 \leq t \leq 2.5$ on the axes below.



(11 marks)

a)

$$y = t(t^2 - 4) = 0$$

$$t = 0, 2, -2$$

$$t = 0$$

$$x = -4, y = 0$$

$$t = \pm 2$$

$$x = 4 - 4 = 0$$

$$y = 0$$

$t = 2$ and $t = -2$ both give O , so C passes through O twice

b)

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3t^2 - 4$$

$$\frac{dy}{dx} = \frac{3t^2 - 4}{2t}$$

$$t = 2$$

$$\frac{dy}{dx} = \frac{12 - 4}{4} = 2$$

$$t = -2$$

$$\frac{dy}{dx} = \frac{12 - 4}{-4} = -2$$

2 and -2

c)

$$y = t(t^2 - 4)$$

$$t^2 - 4 = x$$

$$y = tx$$

$$y^2 = t^2 x^2$$

$$t^2 = x + 4$$

$$y^2 = x^2(x + 4)$$

shown

d)

$$\text{Parallel to the } x\text{-axis: } \frac{dy}{dt} = 0$$

$$3t^2 - 4 = 0$$

$$t = \pm \sqrt{\frac{4}{3}}$$

$$x = \frac{4}{3} - 4 = -\frac{8}{3}$$

$$y = t(t^2 - 4) = t \times \left(-\frac{8}{3}\right)$$

$$t = \sqrt{\frac{2}{3}}$$

$$y = -\frac{16}{3\sqrt{3}} = -\frac{16\sqrt{3}}{9}$$

$$t = -\sqrt{\frac{2}{3}}$$

$$y = \frac{16\sqrt{3}}{9}$$

$$\left(-\frac{8}{3}, \frac{16\sqrt{3}}{9}\right) \text{ and } \left(-\frac{8}{3}, -\frac{16\sqrt{3}}{9}\right)$$

e)

Loop from $(-4, 0)$ back to O , symmetrical in the x -axis

Vertical tangent at $(-4, 0)$; turning points at $x = -\frac{8}{3}$

Branches cross at O with gradients 2 and -2 , then carry on into $x > 0$

see the sketch