



- 1 Prove by contradiction that, for any integer n , if n^2 is even then n is even.

(3 marks)

Assume the opposite

Assume n^2 is even and n is odd

Write the odd number in algebra

$$n = 2k + 1, k \text{ an integer}$$

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

$2k^2 + 2k$ is an integer, so n^2 is odd

This contradicts n^2 being even

So if n^2 is even, n is even

- 2 Prove by contradiction that if a is a rational number and b is an irrational number, then $a + b$ is irrational.

(3 marks)

Assume the opposite

Assume $a + b$ is rational

$$a + b = \frac{p}{q}, p \text{ and } q \text{ integers, } q \neq 0$$

$$a = \frac{m}{n}, m \text{ and } n \text{ integers, } n \neq 0$$

Make b the subject

$$\begin{aligned} b &= \frac{p}{q} - \frac{m}{n} \\ &= \frac{pn - mq}{qn} \end{aligned}$$

$pn - mq$ and qn are integers and $qn \neq 0$, so b is rational

This contradicts b being irrational

So $a + b$ is irrational

3

Prove by contradiction that for all positive real numbers x ,

$$x + \frac{9}{x} \geq 6$$

(3 marks)

Assume the opposite

Assume there is an $x > 0$ with $x + \frac{9}{x} < 6$

Multiply by x (positive, so the sign stays)

$$x^2 + 9 < 6x$$

$$x^2 - 6x + 9 < 0$$

$$(x - 3)^2 < 0$$

A real number squared is never negative

Contradiction

$$\text{So } x + \frac{9}{x} \geq 6 \text{ for all } x > 0$$

4

You may assume that, for any integer n , if n^2 is even then n is even.

Prove by contradiction that $\sqrt{2}$ is irrational.

(5 marks)

Assume the opposite

Assume $\sqrt{2}$ is rational

$$\sqrt{2} = \frac{a}{b}, \text{ } a \text{ and } b \text{ integers with no common factor, } b \neq 0$$

Square both sides

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2b^2$$

So a^2 is even, so a is even

$$a = 2c, \text{ } c \text{ an integer}$$

$$(2c)^2 = 2b^2$$

$$4c^2 = 2b^2$$

$$b^2 = 2c^2$$

So b^2 is even, so b is even

a and b are both even, so they have a common factor 2

This contradicts a and b having no common factor

So $\sqrt{2}$ is irrational

5

Prove by contradiction that there are no integers x and y such that

$$x^2 - y^2 = 10$$

(4 marks)

Assume the opposite

Assume integers x and y exist with $x^2 - y^2 = 10$

Difference of two squares

$$(x - y)(x + y) = 10$$

The two brackets add to an even number

$$(x + y) + (x - y) = 2x$$

So $x - y$ and $x + y$ are both odd or both even

Case 1: both odd

odd $\times$ odd = odd, but 10 is even: contradiction

Case 2: both even

$$(2m)(2n) = 4mn$$

so the product is a multiple of 4, but 10 is not: contradiction

So there are no integers x and y with $x^2 - y^2 = 10$

6

Prove by contradiction that $\log_2 5$ is irrational.

(4 marks)

Assume the opposite

Assume $\log_2 5$ is rational

$$\log_2 5 = \frac{p}{q}, \text{ } p \text{ and } q \text{ integers, } q \neq 0$$

 $\log_2 5 > 0$ since $5 > 1$, so take $p, q > 0$

Undo the log

$$2^{p/q} = 5$$

Raise both sides to the power q

$$2^p = 5^q$$

 $p \geq 1$ so 2^p is even 5^q is a product of odd numbers, so it is odd

Even = odd is impossible: contradiction

So $\log_2 5$ is irrational

7

A student is asked to prove by contradiction that $\sin x + \cos x > 1$ for all values of x with $0^\circ < x < 90^\circ$

The student begins:

Assume that there is an angle x with $0^\circ < x < 90^\circ$ such that $\sin x + \cos x \leq 1$

Both sides are positive, so $(\sin x + \cos x)^2 \leq 1$

Complete the proof.

(3 marks)

Expand the bracket

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x \leq 1$$

Use $\sin^2 x + \cos^2 x = 1$

$$1 + 2 \sin x \cos x \leq 1$$

$$\sin x \cos x \leq 0$$

But for $0^\circ < x < 90^\circ$, $\sin x > 0$ and $\cos x > 0$

so $\sin x \cos x > 0$

Contradiction

So $\sin x + \cos x > 1$ for $0^\circ < x < 90^\circ$

8

Prove by contradiction that there are infinitely many prime numbers.

(5 marks)

Assume the opposite

Assume there is a finite number of primes, $p_1, p_2, p_3, \dots, p_n$

Multiply them all together and add 1

$$N = p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$$

Dividing N by any p_i leaves remainder 1

So none of $p_1, \dots, p_n$ is a factor of N

$N > 1$, so either N is prime, or N has a prime factor

Either way there is a prime that is not in the list

This contradicts the list containing every prime

So there are infinitely many primes

- a. Prove that, for any integer n , if n^2 is a multiple of 3 then n is a multiple of 3
 b. Hence prove by contradiction that $\sqrt{3}$ is irrational.

(7 marks)

a) Assume the opposite

Assume n^2 is a multiple of 3 but n is not

Then $n = 3k + 1$ or $n = 3k + 2$, k an integer

Case $n = 3k + 1$

$$\begin{aligned} n^2 &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

remainder 1, not a multiple of 3

Case $n = 3k + 2$

$$\begin{aligned} n^2 &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

remainder 1, not a multiple of 3

Both cases contradict n^2 being a multiple of 3

So if 3 divides n^2 then 3 divides n

b) Assume the opposite

$$\sqrt{3} = \frac{a}{b}, \text{ } a \text{ and } b \text{ integers with no common factor, } b \neq 0$$

$$3 = \frac{a^2}{b^2}$$

$$a^2 = 3b^2$$

So 3 divides a^2 , so 3 divides a , by part (a)

$$a = 3c, \text{ } c \text{ an integer}$$

$$9c^2 = 3b^2$$

$$b^2 = 3c^2$$

So 3 divides b^2 , so 3 divides b

3 is a common factor of a and b : contradiction

So $\sqrt{3}$ is irrational

- a. Prove that, for any integer p , if p^3 is even then p is even.
 b. Hence prove by contradiction that $\sqrt[3]{2}$ is irrational.
 c. Deduce, using a further proof by contradiction, that $5 - \sqrt[3]{2}$ is irrational.

(8 marks)

a) Assume the opposite

Assume p^3 is even and p is odd

$$p = 2k + 1, k \text{ an integer}$$

$$p^3 = (2k + 1)^3$$

$$= 8k^3 + 12k^2 + 6k + 1$$

$$= 2(4k^3 + 6k^2 + 3k) + 1$$

This is odd: contradiction

So if p^3 is even, p is even

b) Assume the opposite

$$\sqrt[3]{2} = \frac{a}{b}, a \text{ and } b \text{ integers with no common factor, } b \neq 0$$

Cube both sides

$$2 = \frac{a^3}{b^3}$$

$$a^3 = 2b^3$$

So a^3 is even, so a is even, by part (a)

$$a = 2c, c \text{ an integer}$$

$$8c^3 = 2b^3$$

$$b^3 = 4c^3 = 2(2c^3)$$

So b^3 is even, so b is even

a and b have a common factor 2: contradiction

So $\sqrt[3]{2}$ is irrational

c) Assume the opposite

$$5 - \sqrt[3]{2} = \frac{m}{n}, m \text{ and } n \text{ integers, } n \neq 0$$

$$\sqrt[3]{2} = 5 - \frac{m}{n}$$

$$= \frac{5n - m}{n}$$

$5n - m$ and n are integers, so $\sqrt[3]{2}$ would be rational

This contradicts part (b)

So $5 - \sqrt[3]{2}$ is irrational