

Reciprocal trigonometric functions

Worked solutions

A-level Maths topic practice



This topic online

1

Find the exact value of

a. $\sec \frac{3\pi}{4}$

b. $\operatorname{cosec} \frac{\pi}{3}$

c. $\cot \frac{7\pi}{6}$

(3 marks)

a)

$$\sec \frac{3\pi}{4} = \frac{1}{\cos \frac{3\pi}{4}}$$

$$\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \text{ (second quadrant)}$$

$$\begin{aligned} \sec \frac{3\pi}{4} &= 1 \div \left(-\frac{\sqrt{2}}{2}\right) \\ &= -\frac{2}{\sqrt{2}} \end{aligned}$$

$$\boxed{-\sqrt{2}}$$

b)

$$\begin{aligned} \operatorname{cosec} \frac{\pi}{3} &= \frac{1}{\sin \frac{\pi}{3}} \\ &= 1 \div \frac{\sqrt{3}}{2} \\ &= \frac{2}{\sqrt{3}} \end{aligned}$$

$$\boxed{\frac{2\sqrt{3}}{3}}$$

c)

$$\cot \frac{7\pi}{6} = \frac{\cos \frac{7\pi}{6}}{\sin \frac{7\pi}{6}}$$

Third quadrant: sin and cos both negative

$$= \left(-\frac{\sqrt{3}}{2}\right) \div \left(-\frac{1}{2}\right)$$

$$\boxed{\sqrt{3}}$$

2 Solve, for $0 \leq x < 2\pi$, the equation

$$2 \operatorname{cosec} x = 5$$

Give your answers to 3 significant figures.

(3 marks)

$$\operatorname{cosec} x = \frac{5}{2}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\sin x = \frac{2}{5}$$

$$x = \sin^{-1}(0.4)$$

$$= 0.4115\dots$$

Second solution: $\pi - x$

$$x = \pi - 0.4115\dots$$

$$= 2.7300\dots$$

$$x = 0.412, 2.73$$

3 a. For $y = \arcsin x$, state the largest possible domain and the corresponding range.

b. Find the exact value of x for which $\arccos(2x - 1) = \frac{\pi}{3}$

(4 marks)

a)

$$\arcsin \text{ undoes } \sin \text{ on } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\text{domain: } -1 \leq x \leq 1$$

$$\text{range: } -\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$$

$$-1 \leq x \leq 1, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

b)

Take cos of both sides

$$2x - 1 = \cos \frac{\pi}{3}$$

$$2x - 1 = \frac{1}{2}$$

$$2x = \frac{3}{2}$$

$$x = \frac{3}{4}$$

4

Given that $\tan \theta = -2$, where $\frac{\pi}{2} < \theta < \pi$, find the exact value of

- a. $\sec \theta$
b. $\operatorname{cosec} \theta$

(4 marks)

a)

Use $1 + \tan^2 \theta = \sec^2 \theta$

$$\sec^2 \theta = 1 + (-2)^2$$

$$\sec^2 \theta = 5$$

$$\sec \theta = \pm \sqrt{5}$$

Second quadrant: $\cos \theta < 0$, so $\sec \theta < 0$

$$\sec \theta = -\sqrt{5}$$

b)

$$\cot \theta = \frac{1}{\tan \theta} = -\frac{1}{2}$$

Use $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

$$\operatorname{cosec}^2 \theta = 1 + \frac{1}{4}$$

$$\operatorname{cosec}^2 \theta = \frac{5}{4}$$

Second quadrant: $\sin \theta > 0$, so $\operatorname{cosec} \theta > 0$

$$\operatorname{cosec} \theta = \frac{\sqrt{5}}{2}$$

5

Prove that

$$\sec^2 \theta + \operatorname{cosec}^2 \theta \equiv \sec^2 \theta \operatorname{cosec}^2 \theta$$

(3 marks)

Start with the left-hand side

$$\text{LHS} = \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}$$

Common denominator

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta \cos^2 \theta}$$

Use $\sin^2 \theta + \cos^2 \theta = 1$

$$= \frac{1}{\sin^2 \theta \cos^2 \theta}$$

$$= \frac{1}{\cos^2 \theta} \times \frac{1}{\sin^2 \theta}$$

$$= \sec^2 \theta \operatorname{cosec}^2 \theta$$

$$\boxed{\text{LHS} = \text{RHS}}$$

6

Solve, for $0 \leq x < 360^\circ$, the equation

$$\tan^2 x = 2 \sec x + 2$$

Give your answers to 1 decimal place where appropriate.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5 marks)

$$\text{Use } \tan^2 x = \sec^2 x - 1$$

$$\sec^2 x - 1 = 2 \sec x + 2$$

$$\sec^2 x - 2 \sec x - 3 = 0$$

$$(\sec x - 3)(\sec x + 1) = 0$$

$$\sec x = 3 \text{ or } \sec x = -1$$

$$\sec x = 3$$

$$\cos x = \frac{1}{3}$$

$$x = \cos^{-1}\left(\frac{1}{3}\right) = 70.52...^\circ$$

$$x = 360^\circ - 70.52...^\circ = 289.47...^\circ$$

$$\sec x = -1$$

$$\cos x = -1$$

$$x = 180^\circ$$

$$x = 70.5^\circ, 180^\circ, 289.5^\circ$$

7

Solve, for $0 \leq \theta < 2\pi$, the equation

$$3 \cot^2 \theta - 5 \operatorname{cosec} \theta + 1 = 0$$

Give your answers as exact multiples of π .

(5 marks)

$$\text{Use } \cot^2 \theta = \operatorname{cosec}^2 \theta - 1$$

$$3(\operatorname{cosec}^2 \theta - 1) - 5 \operatorname{cosec} \theta + 1 = 0$$

$$3 \operatorname{cosec}^2 \theta - 5 \operatorname{cosec} \theta - 2 = 0$$

$$(3 \operatorname{cosec} \theta + 1)(\operatorname{cosec} \theta - 2) = 0$$

$$\operatorname{cosec} \theta = -\frac{1}{3} \text{ or } \operatorname{cosec} \theta = 2$$

$$|\operatorname{cosec} \theta| \geq 1, \text{ so } \operatorname{cosec} \theta = -\frac{1}{3} \text{ has no solutions}$$

$$\operatorname{cosec} \theta = 2$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$$

8

Solve, for $0 \leq x < 2\pi$, the equation

$$\operatorname{cosec} x = 4 \cos x$$

Give your answers as exact multiples of π .

(4 marks)

$$\frac{1}{\sin x} = 4 \cos x$$

$$1 = 4 \sin x \cos x$$

Use $\sin 2x = 2 \sin x \cos x$

$$1 = 2 \sin 2x$$

$$\sin 2x = \frac{1}{2}$$

Double the interval

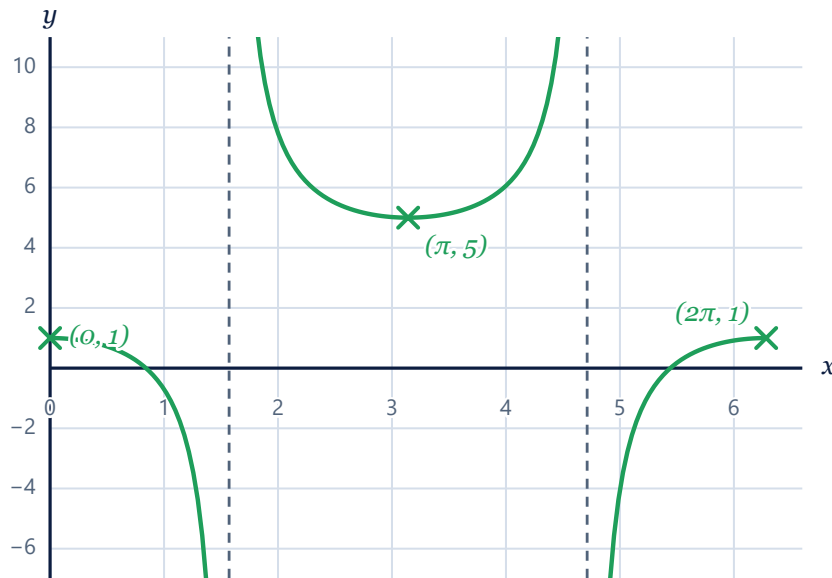
$$0 \leq 2x < 4\pi$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

9

- a. Sketch the graph of $y = 3 - 2 \sec x$ for $0 \leq x \leq 2\pi$. Show the coordinates of the end points and of any turning points, and the equations of any asymptotes.
- b. Hence state the set of values of k for which the equation $3 - 2 \sec x = k$ has no solutions.



(6 marks)

a)

Asymptotes where $\cos x = 0$

$$x = \frac{\pi}{2}, x = \frac{3\pi}{2}$$

$$x = 0: y = 3 - 2(1) = 1$$

$$x = 2\pi: y = 3 - 2(1) = 1$$

$$x = \pi: y = 3 - 2(-1) = 5$$

Outer pieces: $\sec x \geq 1$, so $y \leq 1$ (curve heads down to the asymptotes)Middle piece: $\sec x \leq -1$, so $y \geq 5$ (U shape, minimum $(\pi, 5)$)

Sketch: asymptotes $x = \frac{\pi}{2}, \frac{3\pi}{2}$; $(0, 1), (2\pi, 1)$; min $(\pi, 5)$

b)

Range of $y = 3 - 2 \sec x$ is $y \leq 1$ or $y \geq 5$

$$1 < k < 5$$

10

The function f is defined by

$$f(x) = \pi - 2 \arccos\left(\frac{x}{3}\right)$$

- State the largest possible domain of f .
- Find the range of f .
- Solve the equation $f(x) = \frac{\pi}{3}$

(5 marks)

a)

arccos needs an input between -1 and 1

$$-1 \leq \frac{x}{3} \leq 1$$

$$\boxed{-3 \leq x \leq 3}$$

b)

$$0 \leq \arccos\left(\frac{x}{3}\right) \leq \pi$$

$$0 \leq 2 \arccos\left(\frac{x}{3}\right) \leq 2\pi$$

$$\pi - 2\pi \leq f(x) \leq \pi - 0$$

$$\boxed{-\pi \leq f(x) \leq \pi}$$

c)

$$\pi - 2 \arccos\left(\frac{x}{3}\right) = \frac{\pi}{3}$$

$$2 \arccos\left(\frac{x}{3}\right) = \frac{2\pi}{3}$$

$$\arccos\left(\frac{x}{3}\right) = \frac{\pi}{3}$$

$$\frac{x}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\boxed{x = \frac{3}{2}}$$

a. Prove that

$$\frac{1 + \sec \theta}{\tan \theta + \sin \theta} \equiv \operatorname{cosec} \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

b. Hence solve, for $0 < x < 2\pi$, the equation

$$\frac{1 + \sec x}{\tan x + \sin x} + \cot^2 x = 5$$

Give exact answers where possible and other answers to 3 significant figures.

(8 marks)

a)

Write everything in sin and cos

$$\begin{aligned} 1 + \sec \theta &= 1 + \frac{1}{\cos \theta} = \frac{\cos \theta + 1}{\cos \theta} \\ \tan \theta + \sin \theta &= \frac{\sin \theta}{\cos \theta} + \sin \theta \\ &= \frac{\sin \theta + \sin \theta \cos \theta}{\cos \theta} \\ &= \frac{\sin \theta(1 + \cos \theta)}{\cos \theta} \\ \text{LHS} &= \frac{1 + \cos \theta}{\cos \theta} \times \frac{\cos \theta}{\sin \theta(1 + \cos \theta)} \end{aligned}$$

Cancel $(1 + \cos \theta)$ and $\cos \theta$

$$\begin{aligned} &= \frac{1}{\sin \theta} \\ &= \operatorname{cosec} \theta \end{aligned}$$

$$\boxed{\text{LHS} = \operatorname{cosec} \theta = \text{RHS}}$$

b)

Use part a)

$$\operatorname{cosec} x + \cot^2 x = 5$$

Use $\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\operatorname{cosec} x + \operatorname{cosec}^2 x - 1 = 5$$

$$\operatorname{cosec}^2 x + \operatorname{cosec} x - 6 = 0$$

$$(\operatorname{cosec} x + 3)(\operatorname{cosec} x - 2) = 0$$

$$\operatorname{cosec} x = 2, \text{ so } \sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\operatorname{cosec} x = -3, \text{ so } \sin x = -\frac{1}{3}$$

$$\sin^{-1}\left(\frac{1}{3}\right) = 0.3398\dots$$

sin negative: third and fourth quadrants

$$x = \pi + 0.3398\dots = 3.4814\dots$$

$$x = 2\pi - 0.3398\dots = 5.9433\dots$$

None is a multiple of $\frac{\pi}{2}$, so all are allowed

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, 3.48, 5.94$$

12

A ladder rests on horizontal ground at A and leans against a vertical wall at B . On its way it rests on the top edge F of a fence. The fence is 2 m tall, parallel to the wall and 1 m in front of it. The ladder makes an angle θ radians with the ground, where $0 < \theta < \frac{\pi}{2}$. The ladder is modelled as a straight rod AB of length L metres that just touches F .

a. Show that $L = 2 \operatorname{cosec} \theta + \sec \theta$

b. Show that $\frac{dL}{d\theta} = 0$ when $\tan^3 \theta = 2$

c. Given that this gives the shortest ladder that can reach the wall over the fence, find this shortest length. Give your answer to 3 significant figures.

(8 marks)

a)

Triangle under AF : height 2 m

$$\sin \theta = \frac{2}{AF}$$

$$AF = \frac{2}{\sin \theta} = 2 \operatorname{cosec} \theta$$

Triangle above F : horizontal 1 m to the wall

$$\cos \theta = \frac{1}{FB}$$

$$FB = \frac{1}{\cos \theta} = \sec \theta$$

$$L = AF + FB = 2 \operatorname{cosec} \theta + \sec \theta$$

$$L = 2 \operatorname{cosec} \theta + \sec \theta$$

b)

$$d/d\theta (\operatorname{cosec} \theta) = -\operatorname{cosec} \theta \cot \theta, d/d\theta (\sec \theta) = \sec \theta \tan \theta$$

$$\frac{dL}{d\theta} = -2 \operatorname{cosec} \theta \cot \theta + \sec \theta \tan \theta$$

$$\text{Set } \frac{dL}{d\theta} = 0$$

$$\sec \theta \tan \theta = 2 \operatorname{cosec} \theta \cot \theta$$

$$\frac{\sin \theta}{\cos^2 \theta} = \frac{2 \cos \theta}{\sin^2 \theta}$$

$$\sin^3 \theta = 2 \cos^3 \theta$$

$$\tan^3 \theta = 2$$

$$\tan^3 \theta = 2$$

c)

$$\tan \theta = \sqrt[3]{2} = 1.2599...$$

$$\theta = \tan^{-1}(1.2599...) = 0.8999...$$

$$L = \frac{2}{\sin(0.8999...)} + \frac{1}{\cos(0.8999...)}$$

$$L = 2.5198... + 1.6420...$$

$$L = 4.1619...$$

$$\boxed{L = 4.16 \text{ m}}$$