



1 The 5th term of an arithmetic sequence is 23 and the 12th term is 51

- Find the first term and the common difference.
- Find the sum of the first 20 terms.

(4 marks)

a)

$$n\text{th term} = a + (n - 1)d$$

$$a + 4d = 23$$

$$a + 11d = 51$$

Subtract

$$7d = 28$$

$$d = 4$$

$$a = 23 - 16$$

$$a = 7, d = 4$$

b)

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_{20} = \frac{20}{2}(2 \times 7 + 19 \times 4)$$

$$= 10(14 + 76)$$

$$S_{20} = 900$$

2

A geometric series has first term 48 and common ratio $\frac{3}{4}$

- Find the exact value of the 6th term.
- Find the sum of the first 10 terms, giving your answer to 3 significant figures.
- Find the sum to infinity.

(5 marks)

a)

$$u_n = ar^{n-1}$$

$$\begin{aligned} u_6 &= 48 \times \left(\frac{3}{4}\right)^5 \\ &= 48 \times \frac{243}{1024} \end{aligned}$$

$$u_6 = \frac{729}{64}$$

b)

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\begin{aligned} S_{10} &= \frac{48(1-0.75^{10})}{1-0.75} \\ &= 192(1-0.05631...) \\ &= 181.18... \end{aligned}$$

$$S_{10} = 181 \text{ (3 s.f.)}$$

c)

$|r| < 1$ so the sum to infinity exists

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} \\ &= \frac{48}{0.25} \end{aligned}$$

$$S_{\infty} = 192$$

3

Find the value of

a. $\sum_{r=1}^{30} (5r - 2)$

b. $\sum_{r=1}^8 3 \times 2^r$

(4 marks)

a)

Write out the first and last terms

$$r = 1: 5 - 2 = 3$$

$$r = 30: 150 - 2 = 148$$

Arithmetic, 30 terms: $S = \frac{n}{2}(\text{first} + \text{last})$

$$S = \frac{30}{2}(3 + 148)$$

$$= 15 \times 151$$

2265

b)

$$r = 1: 3 \times 2 = 6$$

Geometric with $a = 6$, $r = 2$, 8 terms

$$S = \frac{6(2^8 - 1)}{2 - 1}$$

$$= 6 \times 255$$

1530

4

A sequence is defined by

$$u_1 = 2, \quad u_{n+1} = ku_n + 3$$

where k is a positive constant. Given that $u_3 = 23$

a. show that $2k^2 + 3k - 20 = 0$

b. find the value of k

c. find $\sum_{r=1}^4 u_r$

(6 marks)

a)

$$u_2 = 2k + 3$$

$$\begin{aligned} u_3 &= k(2k + 3) + 3 \\ &= 2k^2 + 3k + 3 \end{aligned}$$

$$2k^2 + 3k + 3 = 23$$

$$2k^2 + 3k - 20 = 0$$

b)

$$(2k - 5)(k + 4) = 0$$

$$k = \frac{5}{2} \text{ or } k = -4$$

 k is positive

$$k = \frac{5}{2}$$

c)

$$u_1 = 2$$

$$u_2 = 2.5 \times 2 + 3 = 8$$

$$u_3 = 23$$

$$u_4 = 2.5 \times 23 + 3 = 60.5$$

$$\Sigma = 2 + 8 + 23 + 60.5$$

$$93.5$$

5

A sequence is defined by

$$u_1 = 3, \quad u_{n+1} = \frac{1}{1 - u_n}$$

- Find u_2 , u_3 and u_4
- Describe the behaviour of the sequence.
- Find the value of $\sum_{r=1}^{100} u_r$

(6 marks)

a)

$$u_2 = \frac{1}{1 - 3}$$

$$= -\frac{1}{2}$$

$$u_3 = \frac{1}{1 + \frac{1}{2}}$$

$$= \frac{2}{3}$$

$$u_4 = \frac{1}{1 - \frac{2}{3}}$$

$$u_2 = -\frac{1}{2}, \quad u_3 = \frac{2}{3}, \quad u_4 = 3$$

b)

 $u_4 = u_1$, so the terms repeat

periodic, order 3

c)

One cycle

$$3 - \frac{1}{2} + \frac{2}{3} = \frac{19}{6}$$

100 terms = 33 full cycles + 1 term

$$\sum = 33 \times \frac{19}{6} + u_{100}$$

 $u_{100} = u_1$ because $100 = 3 \times 33 + 1$

$$= \frac{209}{2} + 3$$

$$\frac{215}{2}$$

6

The first three terms of a geometric sequence are

$$k + 3, \quad k - 3, \quad 7 - k$$

where k is a constant.

a. Show that $k^2 - 5k - 6 = 0$

Given that the series converges,

b. find the value of k , giving a reason for your answer,

c. find the sum to infinity of the series.

(6 marks)

a)

Common ratio: second $\div$ first = third $\div$ second

$$\frac{k - 3}{k + 3} = \frac{7 - k}{k - 3}$$

$$(k - 3)^2 = (k + 3)(7 - k)$$

$$k^2 - 6k + 9 = 7k - k^2 + 21 - 3k$$

$$k^2 - 6k + 9 = -k^2 + 4k + 21$$

$$2k^2 - 10k - 12 = 0$$

$$k^2 - 5k - 6 = 0$$

b)

$$(k - 6)(k + 1) = 0$$

$k = -1$: terms 2, -4, 8

$$r = -2$$

$|r| > 1$ so this series diverges

$k = 6$: terms 9, 3, 1

$$r = \frac{1}{3}$$

$$k = 6, \text{ since a convergent series needs } |r| < 1$$

c)

$$S_{\infty} = \frac{9}{1 - \frac{1}{3}}$$

$$S_{\infty} = \frac{27}{2}$$

7

Priya saves money each month. She saves £40 in the first month and each month after that she saves £5 more than the month before.

- Find how much she saves in the 12th month.
- Show that the total she has saved after n months is $\frac{5}{2}n(n + 15)$ pounds.
- Find the month in which her total savings first exceed £3000

(6 marks)

a)

$$u_{12} = 40 + 11 \times 5$$

$$\boxed{£95}$$

b)

$$\begin{aligned} S_n &= \frac{n}{2}(2 \times 40 + (n - 1) \times 5) \\ &= \frac{n}{2}(5n + 75) \end{aligned}$$

$$\boxed{S_n = \frac{5}{2}n(n + 15)}$$

c)

$$\frac{5}{2}n(n + 15) > 3000$$

$$n^2 + 15n > 1200$$

$$n^2 + 15n - 1200 > 0$$

Solve the quadratic equation

$$n = \frac{-15 + \sqrt{(225 + 4800)}}{2}$$

$$= 27.94...$$

n must be a whole number above 27.94...

Check

$$S_{27} = 2.5 \times 27 \times 42 = 2835$$

$$S_{28} = 2.5 \times 28 \times 43 = 3010$$

the 28th month

In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

A bakery made a profit of £20 000 in its first year of trading, Year 1. A model predicts that the profit will increase by 4% each year, so that the profits form a geometric sequence.

- Find the first year in which the predicted profit is more than £30 000
- Find the total predicted profit for the first 15 years, giving your answer to the nearest £1000
- Give one reason why the model may not be suitable in the long term.

(7 marks)

a)

$$u_n = 20000 \times 1.04^{n-1}$$

$$20000 \times 1.04^{n-1} > 30000$$

$$1.04^{n-1} > 1.5$$

Take logs of both sides

$$(n-1) \ln 1.04 > \ln 1.5$$

$$n-1 > \frac{\ln 1.5}{\ln 1.04}$$

$$n-1 > 10.33...$$

$$n > 11.33...$$

Year 12

b)

$$\begin{aligned} S_{15} &= \frac{20000(1.04^{15} - 1)}{1.04 - 1} \\ &= 500000(1.80094... - 1) \\ &= 400471.75... \end{aligned}$$

£400 000 (nearest £1000)

c)

4% growth every year for ever is not realistic

9

A geometric series has second term 48 and sum to infinity 300

a. Show that the common ratio, r , satisfies $25r^2 - 25r + 4 = 0$

b. Given that $r > \frac{1}{2}$, find the first term of the series.

c. Find the smallest value of n for which the sum of the first n terms is greater than 299

(8 marks)

a)

$$ar = 48$$

$$\frac{a}{1-r} = 300$$

$$a = 300(1-r)$$

$$300(1-r)r = 48$$

$$300r - 300r^2 = 48$$

$$300r^2 - 300r + 48 = 0$$

Divide by 12

$$25r^2 - 25r + 4 = 0$$

b)

$$(5r-1)(5r-4) = 0$$

$$r = 0.2 \text{ or } r = 0.8$$

$$r > \frac{1}{2}$$

$$r = 0.8$$

$$a = \frac{48}{0.8}$$

$$a = 60$$

c)

$$S_n = \frac{60(1-0.8^n)}{0.2}$$

$$= 300(1-0.8^n)$$

$$300(1-0.8^n) > 299$$

$$1-0.8^n > \frac{299}{300}$$

$$0.8^n < \frac{1}{300}$$

$$n \ln 0.8 < \ln\left(\frac{1}{300}\right)$$

$\ln 0.8$ is negative, so the inequality turns round

$$n > \frac{\ln\left(\frac{1}{300}\right)}{\ln 0.8}$$

$$n > 25.56\dots$$

$$\boxed{n = 26}$$

10

A geometric series has first term a and common ratio r , where $r \neq 1$

a. Prove that the sum of the first n terms of the series is $\frac{a(1-r^n)}{1-r}$

b. Hence, or otherwise, show that $\sum_{n=1}^{\infty} 12 \cos^{2n} \left(\frac{\pi}{6} \right) = 36$

(7 marks)

a)

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

Multiply by r

$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$$

Subtract: everything cancels except the first and last terms

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$r \neq 1$, so divide by $1-r$

$$S_n = \frac{a(1-r^n)}{1-r}$$

b)

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cos^2 \frac{\pi}{6} = \frac{3}{4}$$

$$12 \cos^{2n} \left(\frac{\pi}{6} \right) = 12 \left(\frac{3}{4} \right)^n$$

$n=1$ gives the first term

$$a = 12 \times \frac{3}{4} = 9$$

$$r = \frac{3}{4}$$

$|r| < 1$, so $r^n \rightarrow 0$ and $S_n \rightarrow \frac{a}{1-r}$

$$S_{\infty} = \frac{9}{1 - \frac{3}{4}}$$

$$= 36$$

a. Given that $\sum_{r=1}^n (4r+1) = 1430$, find the value of n .

b. Find the value of k for which $\sum_{r=1}^k \log_2(8^r) = 315$

c. Show that $\sum_{r=1}^{20} \ln(3^r e^2) = 210 \ln 3 + 40$

(8 marks)

a)

Arithmetic: first term 5, last term $4n+1$

$$\frac{n}{2}(5 + 4n + 1) = 1430$$

$$n(2n + 3) = 1430$$

$$2n^2 + 3n - 1430 = 0$$

$$(2n + 55)(n - 26) = 0$$

n is a positive whole number

$$\boxed{n = 26}$$

b)

Simplify the log first

$$\log_2(8^r) = r \log_2 8$$

$$= 3r$$

$$\sum 3r = 3 \times \frac{k(k+1)}{2}$$

$$\frac{3k(k+1)}{2} = 315$$

$$k(k+1) = 210$$

$$k^2 + k - 210 = 0$$

$$(k + 15)(k - 14) = 0$$

$$\boxed{k = 14}$$

c)

$$\ln(3^r e^2) = \ln 3^r + \ln e^2$$

$$= r \ln 3 + 2$$

$$\sum = \ln 3 \times (1 + 2 + \dots + 20) + 20 \times 2$$

$$= \ln 3 \times \frac{20 \times 21}{2} + 40$$

$$= 210 \ln 3 + 40$$

12

A fishery stocks a lake with fish. A model for the number of fish, P_n , in the lake at the start of year n is

$$P_1 = 500, \quad P_{n+1} = 0.8P_n + 150$$

- Find P_2 and P_3
- Show that $P_n = 750 - 250(0.8)^{n-1}$ satisfies the model.
- Explain why the sequence is increasing, and state the long-term number of fish predicted by the model.
- Find the first year in which the model predicts more than 740 fish.

(10 marks)

a)

$$\begin{aligned} P_2 &= 0.8 \times 500 + 150 \\ &= 550 \end{aligned}$$

$$P_3 = 0.8 \times 550 + 150$$

$$P_2 = 550, P_3 = 590$$

b)

Check the first term

$$\begin{aligned} P_1 &= 750 - 250(0.8)^0 \\ &= 500 \end{aligned}$$

Put the formula into the recurrence

$$\begin{aligned} 0.8P_n + 150 &= 0.8(750 - 250(0.8)^{n-1}) + 150 \\ &= 600 - 250(0.8)^n + 150 \\ &= 750 - 250(0.8)^{n+1-1} \end{aligned}$$

$$= P_{n+1}, \text{ so the formula satisfies the model}$$

c)

$250(0.8)^{n-1}$ is multiplied by 0.8 each year, so it gets smaller

Less is taken from 750 each year, so P_n increases

As $n \rightarrow \infty$, $(0.8)^{n-1} \rightarrow 0$

Long term: 750 fish

d)

$$750 - 250(0.8)^{n-1} > 740$$

$$250(0.8)^{n-1} < 10$$

$$(0.8)^{n-1} < 0.04$$

$$(n-1) \ln 0.8 < \ln 0.04$$

$\ln 0.8 < 0$, so the inequality turns round

$$n - 1 > \frac{\ln 0.04}{\ln 0.8}$$

$$n - 1 > 14.42\dots$$

$$n > 15.42\dots$$

Check: $p_{15} = 739.0$, $p_{16} = 741.2$

year 16