

Addition formulae, double angles and $R \sin(x + a)$

Worked solutions

A-level Maths topic practice



This topic online

1

Without using a calculator, find the exact value of

a. $\sin \frac{7\pi}{12}$, giving your answer in the form $\frac{\sqrt{a} + \sqrt{b}}{4}$

b. $\tan \frac{\pi}{12}$, giving your answer in the form $p + q\sqrt{3}$ where p and q are integers.

(4 marks)

a)

Split into angles with known values

$$\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\begin{aligned} \sin \frac{7\pi}{12} &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} + \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \end{aligned}$$

$$\boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}$$

b)

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\begin{aligned} \tan \frac{\pi}{12} &= \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}} \\ &= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \end{aligned}$$

Rationalise: multiply top and bottom by $\sqrt{3} - 1$

$$\begin{aligned} &= \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \\ &= \frac{3 - 2\sqrt{3} + 1}{3 - 1} \\ &= \frac{4 - 2\sqrt{3}}{2} \end{aligned}$$

$$\boxed{2 - \sqrt{3}}$$

2

Given that $\sin A = \frac{4}{5}$, where $\frac{\pi}{2} < A < \pi$, and $\cos B = \frac{5}{13}$, where $0 < B < \frac{\pi}{2}$, find the exact value of

a. $\sin 2A$

b. $\cos(A - B)$

(5 marks)

a)

Use $\sin^2 A + \cos^2 A = 1$

$$\begin{aligned}\cos^2 A &= 1 - \frac{16}{25} \\ &= \frac{9}{25}\end{aligned}$$

A is in the second quadrant, so $\cos A < 0$

$$\cos A = -\frac{3}{5}$$

$\sin 2A = 2 \sin A \cos A$

$$\sin 2A = 2 \times \frac{4}{5} \times \left(-\frac{3}{5}\right)$$

$$\boxed{\sin 2A = -\frac{24}{25}}$$

b)

$$\begin{aligned}\sin^2 B &= 1 - \frac{25}{169} \\ &= \frac{144}{169}\end{aligned}$$

B is acute, so $\sin B > 0$

$$\sin B = \frac{12}{13}$$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\begin{aligned}\cos(A - B) &= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) \\ &= -\frac{15}{65} + \frac{48}{65}\end{aligned}$$

$$\boxed{\cos(A - B) = \frac{33}{65}}$$

3

- a. Express $5 \sin x - 2 \cos x$ in the form $R \sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

- b. Write down the maximum value of $5 \sin x - 2 \cos x$

- c. Find the smallest positive value of x at which this maximum occurs, giving your answer to 3 significant figures.

(5 marks)

a)

Expand $R \sin(x - \alpha)$

$$R \sin(x - \alpha) = R \sin x \cos \alpha - R \cos x \sin \alpha$$

Compare coefficients

$$R \cos \alpha = 5$$

$$R \sin \alpha = 2$$

Square and add

$$R^2 = 5^2 + 2^2$$

$$R = \sqrt{29}$$

Divide

$$\tan \alpha = \frac{2}{5}$$

$$\alpha = 0.3805...$$

$$\boxed{\sqrt{29} \sin(x - 0.381)}$$

b)

The maximum of $\sin(x - \alpha)$ is 1

$$\boxed{\sqrt{29}}$$

c)

$$x - 0.3805... = \frac{\pi}{2}$$

$$x = 1.5707... + 0.3805...$$

$$= 1.9513...$$

$$\boxed{x = 1.95}$$

4

Prove that

$$\cot \theta - \tan \theta \equiv 2 \cot 2\theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

(3 marks)

Write in terms of sin and cos

$$\text{LHS} = \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}$$

Common denominator

$$= \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

$$= \cos 2\theta \div \left(\frac{1}{2} \sin 2\theta\right)$$

$$= \frac{2 \cos 2\theta}{\sin 2\theta}$$

$$= 2 \cot 2\theta$$

$= \text{RHS, so the identity is proved}$

5

In this question you must show all stages of your working.

Solve, for $0 \leq x < 2\pi$, the equation

$$\cos 2x = 3 \sin x - 1$$

Give your answers as exact multiples of π .

(5 marks)

Use $\cos 2x = 1 - 2 \sin^2 x$

$$1 - 2 \sin^2 x = 3 \sin x - 1$$

$$0 = 2 \sin^2 x + 3 \sin x - 2$$

Factorise the quadratic in $\sin x$

$$(2 \sin x - 1)(\sin x + 2) = 0$$

$$\sin x = \frac{1}{2}$$

$$\sin x = -2$$

$\sin x = -2$ has no solutions, since $-1 \leq \sin x \leq 1$

$$x = \frac{\pi}{6}$$

Second solution: $\pi - x$

$$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

6

Solve, for $0 \leq \theta < 360^\circ$, the equation

$$5 \sin 2\theta = 3 \cos \theta$$

Give your answers to one decimal place where appropriate.

(5 marks)

$$\text{Use } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$10 \sin \theta \cos \theta = 3 \cos \theta$$

$$10 \sin \theta \cos \theta - 3 \cos \theta = 0$$

Factorise (dividing by $\cos \theta$ would lose solutions)

$$\cos \theta (10 \sin \theta - 3) = 0$$

$$\cos \theta = 0$$

$$\theta = 90^\circ, 270^\circ$$

$$\sin \theta = 0.3$$

$$\theta = \sin^{-1}(0.3)$$

$$= 17.457^\circ$$

$$\theta = 180^\circ - 17.457^\circ$$

$$= 162.54^\circ$$

$$\theta = 17.5^\circ, 90^\circ, 162.5^\circ, 270^\circ$$

7

Solve, for $0 \leq \theta < \pi$, the equation

$$\tan\left(\theta + \frac{\pi}{4}\right) = 6 \tan \theta$$

Give your answers in radians to 3 decimal places.

(5 marks)

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \tan \frac{\pi}{4} = 1$$

$$\frac{\tan \theta + 1}{1 - \tan \theta} = 6 \tan \theta$$

$$\text{Let } t = \tan \theta$$

$$t + 1 = 6t(1 - t)$$

$$t + 1 = 6t - 6t^2$$

$$6t^2 - 5t + 1 = 0$$

$$(3t - 1)(2t - 1) = 0$$

$$t = \frac{1}{3}$$

$$t = \frac{1}{2}$$

$$\tan \theta = \frac{1}{3}$$

$$\theta = 0.32175\dots$$

$$\tan \theta = \frac{1}{2}$$

$$\theta = 0.46364\dots$$

$\tan$ has period π , so no more solutions in $0 \leq \theta < \pi$

$$\theta = 0.322, 0.464$$

- a. Show that $\sin 3x \equiv 3 \sin x - 4 \sin^3 x$
 b. Hence solve, for $0 < x < \pi$, the equation
 $\sin 3x = 2 \sin x$
 giving your answers as exact multiples of π .

(7 marks)

a)

Write $3x$ as $2x + x$

$$\begin{aligned}\sin 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x\end{aligned}$$

Double angle formulae

$$\begin{aligned}&= 2 \sin x \cos x \times \cos x + (1 - 2 \sin^2 x) \sin x \\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x\end{aligned}$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\begin{aligned}&= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

shown

b)

$$3 \sin x - 4 \sin^3 x = 2 \sin x$$

$$\sin x - 4 \sin^3 x = 0$$

$$\sin x (1 - 4 \sin^2 x) = 0$$

 $\sin x = 0$ gives $x = 0$ or π , not in $0 < x < \pi$

$$\sin^2 x = \frac{1}{4}$$

$$\sin x = \pm \frac{1}{2}$$

 $\sin x > 0$ for $0 < x < \pi$, so only $\sin x = \frac{1}{2}$

$$x = \frac{\pi}{6}$$

$$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

- a. Express $2 \sin \theta + 5 \cos \theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$.
Give the exact value of R and give the value of α to 2 decimal places.
- b. Hence solve, for $0 \leq \theta < 360^\circ$, the equation
 $2 \sin \theta + 5 \cos \theta = 3$
giving your answers to one decimal place.

(6 marks)

a)

$$R \sin(\theta + \alpha) = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$$

Compare coefficients

$$R \cos \alpha = 2$$

$$R \sin \alpha = 5$$

$$R^2 = 2^2 + 5^2 = 29$$

$$R = \sqrt{29}$$

$$\tan \alpha = \frac{5}{2}$$

$$\alpha = 68.198...^\circ$$

$$\boxed{\sqrt{29} \sin(\theta + 68.20^\circ)}$$

b)

$$\sqrt{29} \sin(\theta + 68.198...^\circ) = 3$$

$$\sin(\theta + 68.198...^\circ) = \frac{3}{\sqrt{29}}$$

$$= 0.55708...$$

New range: $68.20^\circ \leq \theta + \alpha < 428.20^\circ$

$$\sin^{-1}(0.55708...) = 33.854...^\circ$$

 33.85° is below the range, so use $180^\circ - 33.85^\circ$ and $360^\circ + 33.85^\circ$

$$\theta + 68.198...^\circ = 146.145...^\circ$$

$$\theta + 68.198...^\circ = 393.854...^\circ$$

$$\theta = 77.946...^\circ$$

$$\theta = 325.655...^\circ$$

$$\boxed{\theta = 77.9^\circ, 325.7^\circ}$$

In this question you must show all stages of your working.

a. Express $\sqrt{3} \sin x + \cos x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving the exact values of R and α .

b. Hence solve, for $0 \leq x < \pi$, the equation

$$\sqrt{3} \sin 2x + \cos 2x = 1$$

c. Find the maximum value and the minimum value of

$$\frac{1}{4 + \sqrt{3} \sin x + \cos x}$$

(8 marks)

a)

$$R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$$

$$R \cos \alpha = \sqrt{3}$$

$$R \sin \alpha = 1$$

$$R^2 = 3 + 1 = 4$$

$$R = 2$$

$$\tan \alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$\boxed{2 \sin\left(x + \frac{\pi}{6}\right)}$$

b)

Same expression with $2x$ in place of x

$$2 \sin\left(2x + \frac{\pi}{6}\right) = 1$$

$$\sin\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\text{Range: } \frac{\pi}{6} \leq 2x + \frac{\pi}{6} < \frac{13\pi}{6}$$

$$2x + \frac{\pi}{6} = \frac{\pi}{6}$$

$$2x + \frac{\pi}{6} = \frac{5\pi}{6}$$

$\frac{13\pi}{6}$ is not included

$$2x = 0$$

$$2x = \frac{2\pi}{3}$$

$$x = 0, \frac{\pi}{3}$$

c)

$$-1 \leq \sin\left(x + \frac{\pi}{6}\right) \leq 1$$

Least value of the denominator

$$4 + 2 \sin\left(x + \frac{\pi}{6}\right) = 4 - 2 = 2$$

Greatest value of the denominator

$$4 + 2 \sin\left(x + \frac{\pi}{6}\right) = 4 + 2 = 6$$

Smallest denominator gives the largest value

$$\text{maximum} = \frac{1}{2}$$

$$\text{minimum} = \frac{1}{6}$$

$$\text{maximum } \frac{1}{2}, \text{ minimum } \frac{1}{6}$$

- 11 The depth of water, D metres, at the entrance to a harbour is modelled by the equation

$$D = 5 + 1.2 \sin\left(\frac{\pi t}{6}\right) - 0.9 \cos\left(\frac{\pi t}{6}\right), \quad 0 \leq t < 12$$

where t is the time in hours after midnight and the angles are in radians.

- a. Express $1.2 \sin x - 0.9 \cos x$ in the form $R \sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of α to 4 decimal places.

- b. Using your answer to part (a), find the maximum depth of the water according to the model.

- c. Find the time when this maximum depth occurs, giving your answer to the nearest minute.

- d. A fishing boat can only pass through the entrance when the depth of water is at least 5.9 metres.

Find, according to the model, the times between which the boat can pass through the entrance, giving your answers to the nearest minute.

(9 marks)

a)

$$R \sin(x - \alpha) = R \sin x \cos \alpha - R \cos x \sin \alpha$$

$$R \cos \alpha = 1.2$$

$$R \sin \alpha = 0.9$$

$$R^2 = 1.2^2 + 0.9^2 = 2.25$$

$$R = 1.5$$

$$\tan \alpha = \frac{0.9}{1.2} = 0.75$$

$$\alpha = 0.64350\dots$$

$$\boxed{1.5 \sin(x - 0.6435)}$$

b)

$$0 = 5 + 1.5 \sin\left(\frac{\pi t}{6} - 0.6435\right)$$

Maximum when $\sin(\dots) = 1$

$$0_{\{max\}} = 5 + 1.5$$

$$\boxed{6.5 \text{ m}}$$

c)

$$\frac{\pi t}{6} - 0.64350\dots = \frac{\pi}{2}$$

$$\frac{\pi t}{6} = 2.21429\dots$$

$$t = \frac{6 \times 2.21429\dots}{\pi}$$

$$= 4.22899\dots \text{ hours}$$

$$0.22899\dots \times 60 = 13.7 \text{ minutes}$$

04:14

d.)

$$5 + 1.5 \sin\left(\frac{\pi t}{6} - \alpha\right) \geq 5.9$$

$$\sin\left(\frac{\pi t}{6} - \alpha\right) \geq 0.6$$

$$\sin^{-1}(0.6) = 0.64350... \text{ (the same as } \alpha \text{)}$$

$$\text{Range: } -\alpha \leq \frac{\pi t}{6} - \alpha < 2\pi - \alpha$$

$$0.64350... \leq \frac{\pi t}{6} - \alpha \leq \pi - 0.64350...$$

$$1.28700... \leq \frac{\pi t}{6} \leq \pi$$

$$\frac{6 \times 1.28700...}{\pi} \leq t \leq 6$$

$$2.45799... \leq t \leq 6$$

$$0.45799... \times 60 = 27.5 \text{ minutes}$$

from 02:27 to 06:00

12

$$f(x) = 4 \sin^2 x + 6 \sin x \cos x, \quad 0 \leq x \leq \pi$$

a. Show that $f(x) = 2 + 3 \sin 2x - 2 \cos 2x$

b. Express $3 \sin 2x - 2 \cos 2x$ in the form $R \sin(2x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 4 decimal places.

c. Hence state the maximum value of $f(x)$ and find the value of x at which it occurs, giving x to 3 significant figures.

d. Find the set of values of the constant k for which the equation $f(x) = k$ has exactly two solutions in the interval $0 \leq x \leq \pi$

(10 marks)

a)

$$\cos 2x = 1 - 2 \sin^2 x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$4 \sin^2 x = 2 - 2 \cos 2x$$

$$\sin 2x = 2 \sin x \cos x$$

$$6 \sin x \cos x = 3 \sin 2x$$

$$f(x) = 2 + 3 \sin 2x - 2 \cos 2x$$

shown

b)

$$R \sin(2x - \alpha) = R \sin 2x \cos \alpha - R \cos 2x \sin \alpha$$

$$R \cos \alpha = 3$$

$$R \sin \alpha = 2$$

$$R = \sqrt{9 + 4} = \sqrt{13}$$

$$\tan \alpha = \frac{2}{3}$$

$$\alpha = 0.58800\dots$$

$$\sqrt{13} \sin(2x - 0.5880)$$

c)

$$f(x) = 2 + \sqrt{13} \sin(2x - \alpha)$$

Maximum when $\sin(2x - \alpha) = 1$

$$f_{\text{max}} = 2 + \sqrt{13}$$

$$2x - 0.58800\dots = \frac{\pi}{2}$$

$$2x = 2.15879\dots$$

$$x = 1.07939\dots$$

maximum $2 + \sqrt{13}$ at $x = 1.08$

d)

Let $u = 2x - \alpha$

$0 \leq x \leq \pi$ gives $-\alpha \leq u \leq 2\pi - \alpha$: one full cycle

$$\sin(-\alpha) = -\frac{2}{\sqrt{13}}$$

$$f(0) = 2 + \sqrt{13} \times \left(-\frac{2}{\sqrt{13}}\right) = 0$$

$$f(\pi) = 0$$

f rises from 0 to $2 + \sqrt{13}$, falls to $2 - \sqrt{13}$, rises back to 0

$k = 2 + \sqrt{13}$ or $k = 2 - \sqrt{13}$: one solution

$k = 0$: three solutions ($x = 0$, $x = \pi$ and one in between)

any other k between $2 - \sqrt{13}$ and $2 + \sqrt{13}$: two solutions

$$2 - \sqrt{13} < k < 2 + \sqrt{13}, k \neq 0$$