



1

The vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

- Find $|\mathbf{a}|$
- Find a unit vector in the direction of $\mathbf{a}$
- Find the angle that $\mathbf{a}$ makes with the positive x-axis, giving your answer in degrees to 1 decimal place.

(4 marks)

a) Square, add, square root

$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + (-3)^2 + 6^2} \\ &= \sqrt{4 + 9 + 36} \\ &= \sqrt{49} \end{aligned}$$

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b) Divide by the magnitude

$$\frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$$

c) $\cos \theta = \frac{\text{x component}}{\text{magnitude}}$

$$\cos \theta = \frac{2}{7}$$

$$\begin{aligned} \theta &= \cos^{-1} \frac{2}{7} \\ &= 73.398...^\circ \end{aligned}$$

73.4°

2

Relative to a fixed origin O , the point A has coordinates $(3, -1, 4)$ and the point B has coordinates $(-1, 2, 16)$.

- Find $\vec{AB}$ in the form $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$
- Find the distance AB .
- Find the coordinates of the midpoint of AB .

(3 marks)

$$a) \rightarrow \vec{AB} = \mathbf{b} - \mathbf{a}$$

$$\rightarrow \vec{AB} = (-1 - 3)\mathbf{i} + (2 - (-1))\mathbf{j} + (16 - 4)\mathbf{k}$$

$$\boxed{-4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}}$$

$$b) \text{ Magnitude of } \rightarrow \vec{AB}$$

$$AB = \sqrt{(-4)^2 + 3^2 + 12^2}$$

$$= \sqrt{169}$$

$$\boxed{13}$$

$$c) \text{ Average the coordinates}$$

$$M = \left(\frac{3 + (-1)}{2}, \frac{-1 + 2}{2}, \frac{4 + 16}{2} \right)$$

$$\boxed{\left(1, \frac{1}{2}, 10 \right)}$$

3

The vectors $\mathbf{p} = 3\mathbf{i} + a\mathbf{j} - 6\mathbf{k}$ and $\mathbf{q} = b\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$, where a and b are constants, are parallel.

Find the value of a and the value of b .

(3 marks)

$$\text{Parallel, so } \mathbf{q} = \lambda \mathbf{p}$$

$$\text{Compare k components (both known)}$$

$$8 = -6\lambda$$

$$\lambda = -\frac{4}{3}$$

$$\text{Compare i components}$$

$$b = 3\lambda$$

$$b = -4$$

$$\text{Compare j components}$$

$$-4 = a\lambda$$

$$-4 = -\frac{4}{3}a$$

$$a = 3$$

$$\boxed{a = 3, b = -4}$$

4

Relative to a fixed origin O , the point A has position vector $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and the point B has position vector $p\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, where p is a positive constant.

Given that $|\vec{AB}| = 7$, find the exact value of p .

(4 marks)

$$\rightarrow \vec{AB} = \mathbf{b} - \mathbf{a}$$

$$= (p - 4)\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$$

Use $|\rightarrow \vec{AB}|^2 = 49$

$$(p - 4)^2 + 5^2 + (-2)^2 = 49$$

$$(p - 4)^2 + 29 = 49$$

$$(p - 4)^2 = 20$$

$$p - 4 = \pm\sqrt{20} = \pm 2\sqrt{5}$$

$$p = 4 + 2\sqrt{5} \text{ or } p = 4 - 2\sqrt{5}$$

$2\sqrt{5} \approx 4.47$ so $4 - 2\sqrt{5} < 0$, reject it

$$p = 4 + 2\sqrt{5}$$

5

Relative to a fixed origin O , the points A , B and C have coordinates $(1, 2, -3)$, $(4, 0, 1)$ and $(10, -4, 9)$ respectively.

a. Show that A , B and C are collinear.

b. Find the ratio $AB : BC$

(4 marks)

a) Find two vectors that share a point

$$\rightarrow \vec{AB} = (4 - 1)\mathbf{i} + (0 - 2)\mathbf{j} + (1 - (-3))\mathbf{k}$$

$$= 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$$\rightarrow \vec{BC} = (10 - 4)\mathbf{i} + (-4 - 0)\mathbf{j} + (9 - 1)\mathbf{k}$$

$$= 6\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$$

$$= 2(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 2\rightarrow \vec{AB}$$

$\rightarrow \vec{BC}$ is parallel to $\rightarrow \vec{AB}$ and they share point B

A , B and C are collinear

b) $\rightarrow \vec{BC} = 2\rightarrow \vec{AB}$

$$AB : BC = 1 : 2$$

6

The triangle ABC has

$$\vec{AB} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \vec{BC} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

a. Find $\vec{AC}$ b. Show that angle ABC is a right angle.c. Find the area of triangle ABC .

(5 marks)

a) Go from A to B then B to C

$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$= (2 + 4)\mathbf{i} + (-1 + 4)\mathbf{j} + (2 - 2)\mathbf{k}$$

$$\boxed{6\mathbf{i} + 3\mathbf{j}}$$

b) Find the three lengths

$$AB = \sqrt{4 + 1 + 4} = 3$$

$$BC = \sqrt{16 + 16 + 4} = 6$$

$$AC = \sqrt{36 + 9 + 0} = \sqrt{45}$$

Check Pythagoras

$$AB^2 + BC^2 = 9 + 36 = 45$$

$$AC^2 = 45$$

$$\boxed{AB^2 + BC^2 = AC^2, \text{ so angle } ABC = 90^\circ}$$

c) Right angle at B, so use $\frac{1}{2} \times \text{base} \times \text{height}$

$$\text{Area} = \frac{1}{2} \times 3 \times 6$$

$$\boxed{9}$$

7

Relative to a fixed origin O , the points P , Q and R have position vectors

$$\mathbf{p} = \mathbf{i} + 2\mathbf{k}, \quad \mathbf{q} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}, \quad \mathbf{r} = 5\mathbf{j} + \mathbf{k}$$

- Find the size of angle QPR , giving your answer in degrees to 1 decimal place.
- Find the area of triangle PQR , giving your answer to 3 significant figures.

(6 marks)

a) Find the three side lengths

$$\rightarrow PQ = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$PQ = \sqrt{4 + 16 + 16} = 6$$

$$\rightarrow PR = -\mathbf{i} + 5\mathbf{j} - \mathbf{k}$$

$$PR = \sqrt{1 + 25 + 1} = \sqrt{27}$$

$$\rightarrow QR = -3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

$$QR = \sqrt{9 + 1 + 9} = \sqrt{19}$$

Cosine rule for the angle at P

$$\cos QPR = \frac{PQ^2 + PR^2 - QR^2}{2 \times PQ \times PR}$$

$$= \frac{36 + 27 - 19}{2 \times 6 \times \sqrt{27}}$$

$$= \frac{44}{12\sqrt{27}}$$

$$= 0.70566...$$

$$QPR = 45.117...^\circ$$

$$\boxed{45.1^\circ}$$

b) Area = $\frac{1}{2}ab \sin C$

$$\text{Area} = \frac{1}{2} \times 6 \times \sqrt{27} \times \sin 45.117...^\circ$$

$$= 11.045...$$

$$\boxed{11.0}$$

8

Relative to a fixed origin O , the point A has position vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and the point B has position vector $8\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$

The straight line l passes through A and B .

a. Find $\vec{AB}$

b. Given that the point P lies on l such that $|\vec{AP}| = 2|\vec{BP}|$, find the two possible position vectors of P .

(5 marks)

$$a) \rightarrow \vec{AB} = \mathbf{b} - \mathbf{a}$$

$$\rightarrow \vec{AB} = (8 - 2)\mathbf{i} + (5 + 1)\mathbf{j} + (-4 - 5)\mathbf{k}$$

$$\boxed{6\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}}$$

b) Case 1: P between A and B , $AP : PB = 2 : 1$

$$\rightarrow \vec{AP} = \frac{2}{3} \rightarrow \vec{AB} = 4\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

$$\mathbf{p} = \mathbf{a} + \rightarrow \vec{AP}$$

$$= (2 + 4)\mathbf{i} + (-1 + 4)\mathbf{j} + (5 - 6)\mathbf{k}$$

$$= 6\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Case 2: P beyond B , so $AP = 2BP$ means $AB = BP$

$$\rightarrow \vec{AP} = 2 \rightarrow \vec{AB} = 12\mathbf{i} + 12\mathbf{j} - 18\mathbf{k}$$

$$\mathbf{p} = \mathbf{a} + \rightarrow \vec{AP}$$

$$= 14\mathbf{i} + 11\mathbf{j} - 13\mathbf{k}$$

$$\boxed{6\mathbf{i} + 3\mathbf{j} - \mathbf{k} \text{ or } 14\mathbf{i} + 11\mathbf{j} - 13\mathbf{k}}$$

9

Relative to a fixed origin O ,

- the point A has position vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
- the point B has position vector $7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$
- the point C has position vector $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

a. Find $\vec{AB}$ and $\vec{CB}$ b. Show that the quadrilateral $OABC$ is a trapezium, giving reasons for your answer.c. Find the ratio $OA : CB$

(5 marks)

a) Subtract position vectors

$$\rightarrow AB = b - a = 5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

$$\rightarrow CB = b - c = 4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

$$\rightarrow AB = 5\mathbf{i} + \mathbf{j} + 4\mathbf{k}, \rightarrow CB = 4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

b) Compare opposite sides

$$\rightarrow OA = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$$\rightarrow CB = 2(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 2\rightarrow OA$$

So OA is parallel to CB

$$\rightarrow OC = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

 $\frac{5}{3} \neq \frac{1}{-2}$, so $\rightarrow AB$ is not a multiple of $\rightarrow OC$

One pair of parallel sides only, so $OABC$ is a trapezium

c) $\rightarrow CB = 2\rightarrow OA$

$$OA : CB = 1 : 2$$

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Relative to a fixed origin O , the points A , B and C have coordinates $(2, 4, -1)$, $(5, 3, 4)$ and $(3, -1, 5)$ respectively.

- Show that $OABC$ is a parallelogram.
- Find the size of angle AOC , giving your answer in degrees to 1 decimal place.
- Find the area of the parallelogram $OABC$, giving your answer to 3 significant figures.

(8 marks)

a) Compare opposite sides

$$\rightarrow OA = 2i + 4j - k$$

$$\rightarrow CB = b - c = (5 - 3)i + (3 + 1)j + (4 - 5)k = 2i + 4j - k$$

$$\rightarrow OC = 3i - j + 5k$$

$$\rightarrow AB = b - a = 3i - j + 5k$$

$$\rightarrow OA = \rightarrow CB \text{ and } \rightarrow OC = \rightarrow AB, \text{ so } OABC \text{ is a parallelogram}$$

b) Lengths of triangle AOC

$$OA = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$OC = \sqrt{9 + 1 + 25} = \sqrt{35}$$

$$\rightarrow AC = c - a = i - 5j + 6k$$

$$AC = \sqrt{1 + 25 + 36} = \sqrt{62}$$

Cosine rule

$$\cos AOC = \frac{21 + 35 - 62}{2\sqrt{21}\sqrt{35}}$$

$$= \frac{-6}{2\sqrt{735}}$$

$$= -0.11066...$$

$$AOC = 96.353...^\circ$$

$$\boxed{96.4^\circ}$$

c) Parallelogram = 2 triangles AOC

$$\text{Area} = 2 \times \frac{1}{2} \times \sqrt{21} \times \sqrt{35} \times \sin 96.353...^\circ$$

$$= 26.944...$$

$$\boxed{26.9}$$

11

An adventure park builds a zip wire. Relative to a fixed origin O at the base of the starting tower, the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ point east, north and vertically upwards, and distances are in metres.

The wire is modelled as a straight line from the top of the tower, T , with position vector $12\mathbf{k}$, to the landing point L , with position vector $40\mathbf{i} + 30\mathbf{j} + 2\mathbf{k}$

- Find the length of the wire.
- Find the angle the wire makes with the horizontal, giving your answer in degrees to 1 decimal place.
- A rider is three quarters of the way along the wire from T to L . Find the height of the rider above the ground, which is horizontal.
- Give one reason why the model of the wire as a straight line may not be realistic.

(8 marks)

a) Vector from T to L

$$\begin{aligned}\rightarrow TL &= (40\mathbf{i} + 30\mathbf{j} + 2\mathbf{k}) - 12\mathbf{k} \\ &= 40\mathbf{i} + 30\mathbf{j} - 10\mathbf{k}\end{aligned}$$

$$\begin{aligned}|\rightarrow TL| &= \sqrt{(40)^2 + (30)^2 + (-10)^2} \\ &= \sqrt{2600} \\ &= 50.990...\end{aligned}$$

51.0 m

b) Horizontal part uses $\mathbf{i}$ and $\mathbf{j}$, vertical part uses $\mathbf{k}$

$$\text{horizontal distance} = \sqrt{(40)^2 + (30)^2} = 50$$

$$\text{vertical drop} = 10$$

$$\tan \theta = \frac{10}{50}$$

$$\theta = 11.309...\text{°}$$

11.3°

c) Start at T and go $\frac{3}{4}$ of $\rightarrow TL$

$$\begin{aligned}\text{position} &= 12\mathbf{k} + \frac{3}{4}(40\mathbf{i} + 30\mathbf{j} - 10\mathbf{k}) \\ &= 30\mathbf{i} + 22.5\mathbf{j} + (12 - 7.5)\mathbf{k} \\ &= 30\mathbf{i} + 22.5\mathbf{j} + 4.5\mathbf{k}\end{aligned}$$

Height is the $\mathbf{k}$ component

4.5 m

d) Criticise the model

The wire sags under its weight and the rider's weight

12

Relative to a fixed origin O , the points A , B and C have coordinates $(2, 0, 1)$, $(4, 3, -1)$ and $(a, 5, a)$ respectively, where a is a constant.

Given that $|\vec{AB}| = |\vec{BC}|$,

- show that $a^2 - 3a + 2 = 0$ and hence find the two possible values of a .
- For the smaller value of a , find the coordinates of the point D such that $ABCD$ is a parallelogram, and explain why $ABCD$ is a rhombus.
- For the smaller value of a , find the size of angle ABC , giving your answer in degrees to 1 decimal place.

(9 marks)

a) Find both vectors

$$\rightarrow AB = 2i + 3j - 2k$$

$$|\rightarrow AB|^2 = 4 + 9 + 4 = 17$$

$$\rightarrow BC = (a - 4)i + 2j + (a + 1)k$$

$$\begin{aligned} |\rightarrow BC|^2 &= (a - 4)^2 + 4 + (a + 1)^2 \\ &= a^2 - 8a + 16 + 4 + a^2 + 2a + 1 \\ &= 2a^2 - 6a + 21 \end{aligned}$$

Set the lengths equal

$$2a^2 - 6a + 21 = 17$$

$$2a^2 - 6a + 4 = 0$$

$$a^2 - 3a + 2 = 0 \text{ (shown)}$$

$$(a - 1)(a - 2) = 0$$

$$a = 1 \text{ or } a = 2$$

b) $a = 1$ so C is $(1, 5, 1)$. In $ABCD$, $\rightarrow AD = \rightarrow BC$

$$d = a + \rightarrow BC$$

$$= (2, 0, 1) + (-3, 2, 2)$$

$$= (-1, 2, 3)$$

$$D(-1, 2, 3)$$

AB and BC are adjacent sides and are equal, so all four sides are equal

$ABCD$ is a rhombus

c) Triangle ABC with $AB = BC = \sqrt{17}$

$$\rightarrow AC = c - a = -i + 5j + 0k$$

$$AC^2 = 1 + 25 = 26$$

Cosine rule

$$\begin{aligned}\cos ABC &= \frac{17 + 17 - 26}{2 \times \sqrt{17} \times \sqrt{17}} \\ &= \frac{8}{34}\end{aligned}$$

$$ABC = 76.391\dots^\circ$$

76.4°
