



- 1 $f(x) = 2x^3 + x^2 - 13x + 6$. Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

(2 marks)

Factor theorem: $(x + 3)$ is a factor if $f(-3) = 0$

$$f(-3) = 2(-3)^3 + (-3)^2 - 13(-3) + 6$$

$$f(-3) = -54 + 9 + 39 + 6$$

$$f(-3) = 0$$

$$f(-3) = 0, \text{ so } (x + 3) \text{ is a factor}$$

- 2 Given that $(x + 1)$ is a factor of $x^3 + 2x^2 - 5x - 6$

- divide $x^3 + 2x^2 - 5x - 6$ by $(x + 1)$
- hence factorise $x^3 + 2x^2 - 5x - 6$ completely.

(4 marks)

a) Algebraic long division by $(x + 1)$

$$x^3 \div x = x^2$$

$$x^2(x + 1) = x^3 + x^2$$

subtract, leaving $x^2 - 5x - 6$

$$x^2 \div x = x$$

$$x(x + 1) = x^2 + x$$

subtract, leaving $-6x - 6$

$$-6x \div x = -6$$

$$-6(x + 1) = -6x - 6$$

subtract, leaving 0

$$x^2 + x - 6$$

b) Factorise the quotient

$$x^2 + x - 6 = (x + 3)(x - 2)$$

$$(x + 1)(x + 3)(x - 2)$$

3

$g(x) = x^3 - 4x^2 + kx + 10$, where k is a constant. Given that $(x + 2)$ is a factor of $g(x)$, find the value of k . You must make your method clear.

(3 marks)

Factor theorem: $(x + 2)$ is a factor so $g(-2) = 0$

$$(-2)^3 - 4(-2)^2 + k(-2) + 10 = 0$$

$$-8 - 16 - 2k + 10 = 0$$

$$-14 - 2k = 0$$

+14

$$-2k = 14$$

+14

$\div -2$

$$k = -7$$

$\div -2$

$$k = -7$$

4

$$f(x) = 2x^3 - 5x^2 - 4x + 3$$

- Show that $(x + 1)$ is a factor of $f(x)$.
- Hence factorise $f(x)$ completely.
- Write down all the solutions of $f(x) = 0$

(5 marks)

a) Factor theorem

$$f(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$$

$$f(-1) = -2 - 5 + 4 + 3 = 0$$

$$f(-1) = 0, \text{ so } (x + 1) \text{ is a factor}$$

b) Divide $f(x)$ by $(x + 1)$

$$2x^3 \div x = 2x^2$$

$$2x^2(x + 1) = 2x^3 + 2x^2$$

subtract, leaving $-7x^2 - 4x + 3$

$$-7x^2 \div x = -7x$$

$$-7x(x + 1) = -7x^2 - 7x$$

subtract, leaving $3x + 3$

$$3x \div x = 3$$

$$3(x + 1) = 3x + 3$$

subtract, leaving 0

$$f(x) = (x + 1)(2x^2 - 7x + 3)$$

Factorise the quadratic: $2 \times 3 = 6$, factors -6 and -1

$$f(x) = (x + 1)(2x - 1)(x - 3)$$

$$(x + 1)(2x - 1)(x - 3)$$

c) Set each factor equal to zero

$$x = -1, x = \frac{1}{2}, x = 3$$

5

Given that $x^3 - 7x - 6 \equiv (x + 1)(x^2 + px + q)$, where p and q are constants,

a. find the values of p and q

b. hence factorise $x^3 - 7x - 6$ completely.

(4 marks)

a) Expand the right-hand side

$$\begin{aligned}(x + 1)(x^2 + px + q) &= x^3 + px^2 + qx + x^2 + px + q \\ &= x^3 + (p + 1)x^2 + (p + q)x + q\end{aligned}$$

Compare coefficients with $x^3 + 0x^2 - 7x - 6$

$$x^2: p + 1 = 0 \Rightarrow p = -1$$

$$\text{constant: } q = -6$$

Check x terms

$$p + q = -1 + (-6) = -7$$

$$\boxed{p = -1, q = -6}$$

b) Factorise $x^2 - x - 6$

$$x^2 - x - 6 = (x - 3)(x + 2)$$

$$\boxed{(x + 1)(x - 3)(x + 2)}$$

6

$$f(x) = x^3 - 2x^2 - 5x - 12$$

- Use the factor theorem to show that $(x - 4)$ is a factor of $f(x)$.
- Write $f(x)$ in the form $(x - 4)(x^2 + bx + c)$, where b and c are integers.
- Hence show that the equation $f(x) = 0$ has exactly one real root.

(5 marks)

a) Factor theorem

$$f(4) = 4^3 - 2(4)^2 - 5(4) - 12$$

$$f(4) = 64 - 32 - 20 - 12 = 0$$

$$f(4) = 0, \text{ so } (x - 4) \text{ is a factor}$$

b) Divide by $(x - 4)$

$$x^3 \div x = x^2$$

$$x^2(x - 4) = x^3 - 4x^2$$

subtract, leaving $2x^2 - 5x - 12$

$$2x^2 \div x = 2x$$

$$2x(x - 4) = 2x^2 - 8x$$

subtract, leaving $3x - 12$

$$3x \div x = 3$$

$$3(x - 4) = 3x - 12$$

subtract, leaving 0

$$(x - 4)(x^2 + 2x + 3)$$

c) Discriminant of the quadratic factor

$$b^2 - 4ac = 2^2 - 4(1)(3)$$

$$b^2 - 4ac = 4 - 12 = -8$$

 $-8 < 0$, so $x^2 + 2x + 3 = 0$ has no real roots

$$\text{the only real root is } x = 4$$

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$f(x) = x^3 + ax^2 + bx - 12$, where a and b are constants. Given that $(x - 3)$ and $(x + 1)$ are both factors of $f(x)$,

- find the values of a and b
- factorise $f(x)$ completely.

(6 marks)

a) Factor theorem with $x = 3$

$$\begin{array}{rcl} 27 + 9a + 3b - 12 & = & 0 \\ 9a + 3b & = & -15 \\ \div 3 & & \div 3 \\ 3a + b & = & -5 \end{array}$$

Factor theorem with $x = -1$

$$\begin{aligned} -1 + a - b - 12 &= 0 \\ a - b &= 13 \end{aligned}$$

Add the two equations to eliminate b

$$\begin{aligned} 4a &= 8 \\ a &= 2 \\ b &= -5 - 3(2) = -11 \end{aligned}$$

$$\boxed{a = 2, b = -11}$$

b) Two factors multiply to a quadratic factor

$$(x - 3)(x + 1) = x^2 - 2x - 3$$

The third factor must make x^3 and -12

$$x^3 + 2x^2 - 11x - 12 = (x^2 - 2x - 3)(x + 4)$$

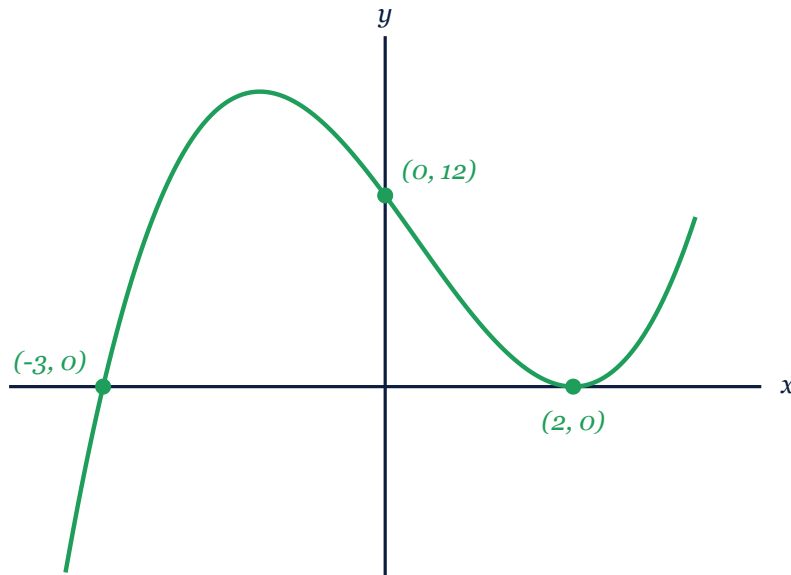
Check the x^2 term: $4x^2 - 2x^2 = 2x^2$

$$\boxed{(x - 3)(x + 1)(x + 4)}$$

8

$$f(x) = x^3 - x^2 - 8x + 12$$

- Show that $(x - 2)$ is a factor of $f(x)$.
- Factorise $f(x)$ completely.
- Sketch the curve with equation $y = f(x)$, showing the coordinates of the points where the curve meets the coordinate axes.



(6 marks)

a) Factor theorem

$$f(2) = 8 - 4 - 16 + 12 = 0$$

$$f(2) = 0, \text{ so } (x - 2) \text{ is a factor}$$

b) Divide by $(x - 2)$

$$x^3 \div x = x^2$$

$$x^2(x - 2) = x^3 - 2x^2$$

subtract, leaving $x^2 - 8x + 12$

$$x^2 \div x = x$$

$$x(x - 2) = x^2 - 2x$$

subtract, leaving $-6x + 12$

$$-6x \div x = -6$$

$$-6(x - 2) = -6x + 12$$

subtract, leaving 0

$$f(x) = (x - 2)(x^2 + x - 6)$$

$$f(x) = (x - 2)(x + 3)(x - 2)$$

$$(x - 2)^2(x + 3)$$

c) Repeated root at 2: the curve touches the x -axis

Single root at -3: crosses; y -intercept $f(0) = 12$

see the sketch

9

$g(x) = 3x^3 - 7x^2 + 2x + 2a$, where a is a positive constant.

Given that $(x - a)$ is a factor of $g(x)$,

a. show that $a(3a - 4)(a - 1) = 0$

b. For the integer value of a , solve $g(x) = 0$, giving your answers in exact form.

(7 marks)

a) Factor theorem: $g(a) = 0$

$$3a^3 - 7a^2 + 2a + 2a = 0$$

$$3a^3 - 7a^2 + 4a = 0$$

$$a(3a^2 - 7a + 4) = 0$$

$$a(3a - 4)(a - 1) = 0$$

shown

b) $a > 0$, so $a = \frac{4}{3}$ or $a = 1$: the integer value is $a = 1$

$$g(x) = 3x^3 - 7x^2 + 2x + 2$$

Divide by $(x - 1)$

$$3x^3 \div x = 3x^2$$

$$3x^2(x - 1) = 3x^3 - 3x^2$$

subtract, leaving $-4x^2 + 2x + 2$

$$-4x^2 \div x = -4x$$

$$-4x(x - 1) = -4x^2 + 4x$$

subtract, leaving $-2x + 2$

$$-2x \div x = -2$$

$$-2(x - 1) = -2x + 2$$

subtract, leaving 0

$$g(x) = (x - 1)(3x^2 - 4x - 2)$$

Quadratic formula on $3x^2 - 4x - 2 = 0$

$$x = \frac{4 \pm \sqrt{(16 + 24)}}{6}$$

$$x = \frac{4 \pm 2\sqrt{10}}{6}$$

$$x = \frac{2 \pm \sqrt{10}}{3}$$

$$x = 1, x = \frac{2 + \sqrt{10}}{3}, x = \frac{2 - \sqrt{10}}{3}$$

10

The curve C has equation $y = f(x)$, where $f(x) = x^3 + px^2 + qx + 6$ and p and q are constants. Given that $(x + 2)$ is a factor of $f(x)$ and that C passes through the point $(2, -4)$,

a. find the values of p and q

b. hence find the coordinates of the points where C meets the x -axis.

(7 marks)

a) Factor theorem: $f(-2) = 0$

$$\begin{array}{rcl} -8 + 4p - 2q + 6 & = & 0 \\ 4p - 2q & = & 2 \\ \div 2 & & \\ 2p - q & = & 1 \quad \div 2 \end{array}$$

Point on the curve: $f(2) = -4$

$$\begin{array}{rcl} 8 + 4p + 2q + 6 & = & -4 \\ 4p + 2q & = & -18 \\ \div 2 & & \\ 2p + q & = & -9 \quad \div 2 \end{array}$$

Add the equations

$$4p = -8$$

$$p = -2$$

$$q = 2p - 1 = -5$$

$$\boxed{p = -2, q = -5}$$

b) Divide $x^3 - 2x^2 - 5x + 6$ by $(x + 2)$

$$x^3 \div x = x^2$$

$$x^2(x + 2) = x^3 + 2x^2$$

subtract, leaving $-4x^2 - 5x + 6$

$$-4x^2 \div x = -4x$$

$$-4x(x + 2) = -4x^2 - 8x$$

subtract, leaving $3x + 6$

$$3x \div x = 3$$

$$3(x + 2) = 3x + 6$$

subtract, leaving 0

$$f(x) = (x + 2)(x^2 - 4x + 3)$$

$$f(x) = (x + 2)(x - 1)(x - 3)$$

$$\boxed{(-2, 0), (1, 0), (3, 0)}$$

11

In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

a. Show that the equation $8^x - 7(4^x) + 14(2^x) - 8 = 0$ can be written as $u^3 - 7u^2 + 14u - 8 = 0$, where $u = 2^x$

b. Hence solve $8^x - 7(4^x) + 14(2^x) - 8 = 0$

(6 marks)

a) Write every power as a power of 2^x

$$8^x = (2^3)^x = (2^x)^3 = u^3$$

$$4^x = (2^2)^x = (2^x)^2 = u^2$$

$$u^3 - 7u^2 + 14u - 8 = 0$$

shown

b) Try small values: $u = 1$

$$1 - 7 + 14 - 8 = 0$$

So $(u - 1)$ is a factor. Divide

$$u^3 \div u = u^2$$

$$u^2(u - 1) = u^3 - u^2$$

subtract, leaving $-6u^2 + 14u - 8$

$$-6u^2 \div u = -6u$$

$$-6u(u - 1) = -6u^2 + 6u$$

subtract, leaving $8u - 8$

$$8u \div u = 8$$

$$8(u - 1) = 8u - 8$$

subtract, leaving 0

$$(u - 1)(u^2 - 6u + 8) = 0$$

$$(u - 1)(u - 2)(u - 4) = 0$$

$$u = 1, 2 \text{ or } 4$$

Change back: $2^x = u$

$$2^x = 1 \Rightarrow x = 0$$

$$2^x = 2 \Rightarrow x = 1$$

$$2^x = 4 \Rightarrow x = 2$$

$$x = 0, x = 1, x = 2$$

12

A joiner in Glasgow makes a wooden storage box in the shape of a closed cuboid. The box has length $(x + 4)$ cm, width x cm and height $(x - 1)$ cm. The volume of the box is 96 cm^3 .

- Show that $x^3 + 3x^2 - 4x - 96 = 0$
- Use the factor theorem to show that $(x - 4)$ is a factor of $x^3 + 3x^2 - 4x - 96$
- Hence show that $x = 4$ is the only real solution of the equation in part (a).
- Find the total surface area of the box.

(8 marks)

a) Volume = Length $\times$ width $\times$ height

$$x(x + 4)(x - 1) = 96$$

$$x(x^2 + 3x - 4) = 96$$

$$x^3 + 3x^2 - 4x - 96 = 0$$

shown

b) Let $f(x) = x^3 + 3x^2 - 4x - 96$

$$f(4) = 64 + 48 - 16 - 96 = 0$$

$f(4) = 0$, so $(x - 4)$ is a factor

c) Divide by $(x - 4)$

$$x^3 \div x = x^2$$

$$x^2(x - 4) = x^3 - 4x^2$$

subtract, leaving $7x^2 - 4x - 96$

$$7x^2 \div x = 7x$$

$$7x(x - 4) = 7x^2 - 28x$$

subtract, leaving $24x - 96$

$$24x \div x = 24$$

$$24(x - 4) = 24x - 96$$

subtract, leaving 0

$$f(x) = (x - 4)(x^2 + 7x + 24)$$

Discriminant of the quadratic factor

$$b^2 - 4ac = 49 - 96 = -47$$

$-47 < 0$, so no more real roots

$x = 4$ is the only real solution

d) Dimensions: 8 cm, 4 cm, 3 cm

$$A = 2(8 \times 4 + 8 \times 3 + 4 \times 3)$$

$$A = 2(32 + 24 + 12)$$

$$A = 136$$

$$136 \text{ cm}^2$$