Differentiation

Worked solutions

A-level Maths topic practice

This topic online

1 Find $\frac{dy}{dx}$ for each of the following.

a. $y = (4x^3 + 1)^5$

b. $y = \ln(5x^2 + 2)$

c. $y = \sec 4x$

(5 marks)

a)

Chain rule: power down, reduce the power, times the derivative of the inside

$$\frac{dy}{dx} = 5(4x^3 + 1)^4 \times 12x^2$$

$$60x^2(4x^3 + 1)^4$$

b)

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

$$\frac{dy}{dx} = \frac{10x}{5x^2 + 2}$$

c)

$\frac{d}{dx} \sec x = \sec x \tan x$, then times the derivative of $4x$

$$\frac{dy}{dx} = 4 \sec 4x \tan 4x$$

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2

a. Given $y = x^3e^{2x}$, show that $\frac{dy}{dx} = x^2e^{2x}(a + bx)$, where a and b are integers to be found.

b. Given $y = \frac{3x - 1}{x^2 + 2}$, find $\frac{dy}{dx}$, giving your answer as a single fraction in its simplest form.

(5 marks)

a)

Product rule: $u = x^3$, $v = e^{2x}$

$$u' = 3x^2$$

$$v' = 2e^{2x}$$

$$\frac{dy}{dx} = u'v + uv'$$

$$= 3x^2 e^{2x} + 2x^3 e^{2x}$$

$$= x^2 e^{2x}(3 + 2x)$$

$$\boxed{a = 3, b = 2}$$

b)

Quotient rule: $u = 3x - 1$, $v = x^2 + 2$

$$u' = 3$$

$$v' = 2x$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{3(x^2 + 2) - 2x(3x - 1)}{(x^2 + 2)^2}$$

$$= \frac{3x^2 + 6 - 6x^2 + 2x}{(x^2 + 2)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{\quad}{(x^2 + 2)^2}}$$

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3

a. The curve with equation $y = 3^{2x}$ passes through the point P where $x = 1$. Find the exact gradient of the curve at P, giving your answer in the form $k \ln 3$

b. Given that $x = \cos y$, where $0 < y < \pi$, show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - x^2)}}$$

(6 marks)

a)

$y = 3^{2x} = 9^x$

$\frac{d}{dx} a^x = a^x \ln a$

$\frac{dy}{dx} = 9^x \ln 9$

At $x = 1$

$\frac{dy}{dx} = 9 \ln 9$

$= 9 \times 2 \ln 3$

$18 \ln 3$

b)

$\frac{dx}{dy} =$

$\frac{dy}{dx} = 1 \div \frac{dx}{dy}$

$\frac{dy}{dx} = -\frac{1}{\sin y}$

$\sin^2 y = 1 - \cos^2 y = 1 - x^2$

$\sin y > 0$ for $0 < y < \pi$, so take the positive root

$\sin y = \sqrt{(1 - x^2)}$

$\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - x^2)}}$

shown

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4 Given that x is measured in radians, prove, from first principles, that the derivative of $\sin 3x$ is $3\cos 3x$

You may assume the formula for $\sin(A \pm B)$ and that, as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5 marks)

Gradient of the chord from x to $x + h$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(3x + 3h) - \sin 3x}{h}$$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{\sin 3x \cos 3h + \cos 3x \sin 3h - \sin 3x}{h}$$

$$= \sin 3x \left(\frac{\cos 3h - 1}{h}\right) + \cos 3x \left(\frac{\sin 3h}{h}\right)$$

Make the limits match: multiply top and bottom by 3

$$= 3\sin 3x \left(\frac{\cos 3h - 1}{3h}\right) + 3\cos 3x \left(\frac{\sin 3h}{3h}\right)$$

As $h \to 0$, $3h \to 0$

$$\frac{\cos 3h - 1}{3h} \to 0$$

$$\frac{\sin 3h}{3h} \to 1$$

$$f'(x) = 3\sin 3x \times 0 + 3\cos 3x \times 1$$

$$= 3\cos 3x$$

proved

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5 The curve C has equation $y = (x^2 - 3)e^x$

a. Find the exact coordinates of the stationary points of C.

b. Use $\frac{d^2y}{dx^2}$ to determine the nature of each stationary point.

(7 marks)

a)

Product rule: $u = x^2 - 3$, $v = e^x$

$$\frac{dy}{dx} = 2x e^x + (x^2 - 3)e^x$$

$$= e^x(x^2 + 2x - 3)$$

$$= e^x(x + 3)(x - 1)$$

$e^x > 0$, so the bracket must be 0

$x = 1$

$x = -3$

$y = (1 - 3)e^1 = -2e$

$y = (9 - 3)e^{-3} = 6e^{-3}$

$\boxed{(1, -2e) \text{ and } (-3, 6e^{-3})}$

b)

Product rule again on $e^x(x^2 + 2x - 3)$

$$\frac{d^2y}{dx^2} = e^x(x^2 + 2x - 3) + e^x(2x + 2)$$

$$= e^x(x^2 + 4x - 1)$$

$x = 1$

$$\frac{d^2y}{dx^2} = 4e > 0$$

$x = -3$

$$\frac{d^2y}{dx^2} = e^{-3}(9 - 12 - 1) = -4e^{-3} < 0$$

$\boxed{(1, -2e) \text{ minimum}, (-3, 6e^{-3}) \text{ maximum}}$

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6 The curve C has equation

$x^2 + 3xy - y^2 = 27$

a. Find $\frac{dy}{dx}$ in terms of x and y.

b. The point $P(3, 3)$ lies on C. Find an equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(6 marks)

a)

Product rule on $3xy$; chain rule on y^2

$$2x + 3y + 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x - 2y) =$$

$$\boxed{\frac{dy}{dx} = -\frac{2x + 3y}{3x - 2y}}$$

b)

$$\frac{dy}{dx} = -\frac{6 + 9}{9 - 6}$$

$$= -$$

Normal gradient $= -1 \div (-5)$

$$m = \frac{1}{5}$$

$$y - 3 = \frac{1}{5}(x - 3)$$

$$5y - 15 = x - 3$$

$$\boxed{x - 5y + 12 = 0}$$

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7 The curve C has equation

$$y = \frac{\sin x}{1 + \cos x}, \quad 0 < x < \pi$$

a. Show that $\frac{dy}{dx} = \frac{1}{1 + \cos x}$

b. Find the exact x coordinate of the point on C where the gradient is $\frac{2}{3}$

(5 marks)

a)

Quotient rule: $u = \sin x$, $v = 1 + \cos x$

$$u' = \cos x$$

$$v' = -\sin x$$

$$\frac{dy}{dx} = \frac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{1 + \cos x}{(1 + \cos x)^2}$$

$$= \frac{1}{1 + \cos x}$$

shown

b)

$$\frac{1}{1 + \cos x} = \frac{2}{3}$$

$$1 + \cos x = \frac{3}{2}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

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8 $f(x) = x^4 - 6x^3 + 12x^2 + 5$

a. Find $f''(x)$

b. Find the set of values of x for which $f(x)$ is concave.

c. Find the coordinates of the points of inflection on the curve $y = f(x)$, and show that neither of them is a stationary point.

(6 marks)

a)

$f'(x) = 4x^3 - 18x^2 + 24x$

$\boxed{f''(x) = 12x^2 - 36x + 24}$

b)

Concave where $f''(x) < 0$

$12(x^2 - 3x + 2) < 0$

$12(x - 1)(x - 2) < 0$

Positive quadratic: negative between the roots

$\boxed{\{x : 1 < x < 2\}}$

c)

$f''(x) = 0$ at $x = 1$ and $x = 2$, and f'' changes sign at each

$f(1) = 1 - 6 + 12 + 5 = 12$

$f(2) = 16 - 48 + 48 + 5 = 21$

Check the gradient there

$f'(1) = 4 - 18 + 24 = 10 \neq 0$

$f'(2) = 32 - 72 + 48 = 8 \neq 0$

$\boxed{(1, 12) \text{ and } (2, 21), \text{ neither stationary}}$

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9 Water is poured into a container in the shape of an inverted right circular cone at a constant rate of 40 $cm^3\ s^{-1}$. When the depth of water is h cm, the radius of the water surface is $\frac{1}{2}h$ cm.

a. Show that the volume of water, $V\ cm^3$, is given by $V = \frac{\pi h^3}{12}$

b. Find the rate at which the depth of the water is increasing at the instant when $h = 8$, giving your answer to 3 significant figures.

c. Find the exact rate at which the area of the water surface is increasing at this instant. (7 marks)

a)

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{\pi h^3}{12}$$

shown

b)

$$\frac{dV}{dh} = \frac{3\pi h^2}{12} = \frac{\pi h^2}{4}$$

At $h = 8$

$$\frac{dV}{dh} = 16\pi$$

Chain rule: $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

$$\frac{dh}{dt} = \frac{40}{16\pi}$$

$$= \frac{5}{2\pi}$$

$$= 0.79577...$$

0.796 cm s^{-1}

c)

$$S = \pi\left(\frac{h}{2}\right)^2 = \frac{\pi h^2}{4}$$

$$\frac{dS}{dh} = \frac{\pi h}{2}$$

$$= 4\pi$$

Chain rule: $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$

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$$\frac{dS}{dt} = 4\pi \times \frac{5}{2\pi}$$

$10\ cm^2\ s^{-1}$

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10 The curve C has equation

$x^2 + xy + y^2 = 12$

a. Find $\frac{dy}{dx}$ in terms of x and y.

b. Find the coordinates of the points on C where the tangent is parallel to the x-axis.

c. Find the coordinates of the points on C where the tangent is parallel to the y-axis. (8 marks)

a)

Product rule on xy; chain rule on y^2

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + 2y) =$$

$$\boxed{\frac{dy}{dx} = -\frac{2x + y}{x + 2y}}$$

b)

$\frac{dy}{dx} = 0$: numerator is 0

$$2x + y = 0$$

$$y =$$

Substitute into the curve

$$x^2 + x(-2x) + (-2x)^2 = 12$$

$$3x^2 = 12$$

$$x = \pm 2$$

$$\boxed{(2, -4) \text{ and } (-2, 4)}$$

c)

Parallel to the y-axis: denominator is 0

$$x + 2y = 0$$

$$x =$$

$$(-2y)^2 + (-2y)y + y^2 = 12$$

$$3y^2 = 12$$

$$y = \pm 2$$

$$\boxed{(-4, 2) \text{ and } (4, -2)}$$

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11 The number of people, N thousand, who have downloaded a new revision app t weeks after its launch is modelled by

$$N = \frac{40}{1 + 7e^{-0.5t}}, \quad t \geq 0$$

a. State the number of downloads at launch and the long-term limit of the number of downloads according to the model.

b. Show that $\frac{dN}{dt} = \frac{N(40 - N)}{80}$

c. Hence find the greatest rate of downloads, in thousands per week, according to the model, and the value of t at which it occurs. Give t to 3 significant figures.

d. Explain why the graph of N against t has a point of inflection at this value of t.

(9 marks)

a)

$t = 0$: $N = \frac{40}{1 + 7} = 5$

As $t \to \infty$, $e \to 0$

$N \to \frac{40}{1} = 40$

5000 at launch; limit 40 000

b)

$N = 40(1 + 7e)^{-1}$

Chain rule

$$\frac{dN}{dt} = = \frac{140e}{(1 + 7e)^2}$$

Now work out the right-hand side

$$40 - N = \frac{40(1 + 7e) - 40}{1 + 7e} = \frac{280e}{1 + 7e}$$

$$\frac{N(40 - N)}{80} = \frac{1}{80} \times \frac{40}{1 + 7e} \times \frac{280e}{1 + 7e} = \frac{140e}{(1 + 7e)^2}$$

shown

c)

Rate $= \frac{1}{80}(40N - N^2)$: a quadratic in N, greatest at $N = 20$ (by symmetry)

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$$\frac{dN}{dt} = \frac{20 \times 20}{80}$$

$$= 5$$

$$\frac{40}{1 + 7e} = 20$$

$$1 + 7e = 2$$

$$e = \frac{1}{7}$$

$$= \ln \frac{1}{7}$$

$$t = 2 \ln 7$$

$$= 3.8918...$$

5 thousand per week at $t = 3.89$

d)

$\frac{dN}{dt}$ increases while $N < 20$ and decreases once $N > 20$

so $\frac{d^2N}{dt^2}$ changes sign from + to - at $t = 2 \ln 7$

convex changes to concave: a point of inflection

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12 The curve C has equation $y = f(x)$, where

$f(x) = \dfrac{\ln x}{x^2}, \quad x > 0$

a. Show that $f'(x) = \dfrac{1 - 2\ln x}{x^3}$

b. Find the exact coordinates of the stationary point of C, and use $f''(x)$ to show that it is a maximum.

c. Find the set of values of x for which C is convex, giving the boundary in the form e^k

d. Given that $f(x) \to 0$ as $x \to \infty$, state, with a reason, the number of solutions of $f(x) = k$ for $0 < k < \dfrac{1}{2e}$

(10 marks)

a)

Quotient rule: $u = \ln x$, $v = x^2$

$u' = \dfrac{1}{x}$

$v' = 2x$

$f'(x) = \dfrac{\frac{1}{x} \times x^2 - 2x \ln x}{x^4}$

$= \dfrac{x - 2x \ln x}{x^4}$

$= \dfrac{1 - 2\ln x}{x^3}$

shown

b)

$1 - 2\ln x = 0$

$\ln x = \dfrac{1}{2}$

$x = e^{1/2} = \sqrt{e}$

$y = \dfrac{\frac{1}{2}}{e} = \dfrac{1}{2e}$

Quotient rule on $f'(x)$: $u = 1 - 2\ln x$, $v = x^3$

$f''(x) = \dfrac{-\frac{2}{x} \times x^3 - 3x^2(1 - 2\ln x)}{x^6}$

$= \dfrac{}{x^6}$

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$$= \frac{6\ln x - 5}{x^4}$$

At $x = \sqrt{e}$, $\ln x = \frac{1}{2}$

$$f''(\sqrt{e}) = \frac{3 - 5}{e^2} = -\frac{2}{e^2} < 0$$

$\left(\sqrt{e}, \frac{1}{2e}\right)$, a maximum

c)

Convex where $f''(x) > 0$, and $x^4 > 0$

$6\ln x - 5 > 0$

$\ln x > \frac{5}{6}$

$\{x : x > e^{5/6}\}$

d)

As $x \to 0$, $f(x) \to -\infty$; $f(1) = 0$

f rises to $\frac{1}{2e}$ at $x = \sqrt{e}$, then falls towards 0 but stays positive

so $y = k$ meets C once on each side of the maximum

two solutions

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