



- 1 The discrete random variable X has the probability distribution

x	1	2	3	4
$P(X = x)$	0.15	0.3	k	0.2

- Find the value of k .
- Find $P(X \geq 2)$.
- Find $P(1 < X < 4)$.

(3 marks)

a) Probabilities add up to 1

$$0.15 + 0.3 + k + 0.2 = 1$$

$$0.65 + k = 1$$

$$k = 0.35$$

$$k = 0.35$$

b) $X \geq 2$ means $X = 2, 3$ or 4

$$P(X \geq 2) = 0.3 + 0.35 + 0.2$$

$$P(X \geq 2) = 0.85$$

$$0.85$$

c) $1 < X < 4$ means $X = 2$ or 3

$$P(1 < X < 4) = 0.3 + 0.35$$

$$P(1 < X < 4) = 0.65$$

$$0.65$$

2

A fair spinner has eight equal sections numbered 1 to 8. The random variable X is the number the spinner lands on.

- Write down the probability distribution of X .
- Find $P(X > 5)$.
- Find the probability that X is a prime number.

(3 marks)

a) Fair, 8 equal sections, so each value is equally likely

$x = 1, 2, 3, 4, 5, 6, 7, 8$

each with probability $\frac{1}{8}$

$$P(X = x) = \frac{1}{8}, x = 1, 2, \dots, 8$$

b) $X > 5$ means $X = 6, 7$ or 8

$$P(X > 5) = 3 \times \frac{1}{8}$$

$$P(X > 5) = \frac{3}{8}$$

$$\frac{3}{8}$$

c) Primes from 1 to 8: 2, 3, 5, 7

$$P(\text{prime}) = 4 \times \frac{1}{8}$$

$$P(\text{prime}) = \frac{4}{8} = \frac{1}{2}$$

$$\frac{1}{2}$$

3

The discrete random variable X has probability function

$$P(X = x) = kx \quad \text{for } x = 1, 2, 3, 4, 5$$

where k is a constant.

- Find the value of k .
- Find $P(X = 4)$.
- Find $P(X \leq 2)$.

(4 marks)

a) Write out each probability

$$x = 1, 2, 3, 4, 5$$

$$\text{probabilities } k, 2k, 3k, 4k, 5k$$

Probabilities add up to 1

$$k + 2k + 3k + 4k + 5k = 1$$

$$15k = 1$$

$$k = \frac{1}{15}$$

$$\boxed{k = \frac{1}{15}}$$

b)

$$P(X = 4) = 4k$$

$$P(X = 4) = \frac{4}{15}$$

$$\boxed{\frac{4}{15}}$$

c) $X \leq 2$ means $X = 1$ or 2

$$P(X \leq 2) = k + 2k$$

$$P(X \leq 2) = 3k$$

$$P(X \leq 2) = \frac{3}{15} = \frac{1}{5}$$

$$\boxed{\frac{1}{5}}$$

4

The discrete random variable X has the probability distribution

x	0	1	2	3	4
$P(X = x)$	0.1	a	0.25	b	0.15

where a and b are constants. Given that $P(X \leq 1) = P(X \geq 3)$

- find the value of a and the value of b
- find $P(1 \leq X < 4)$.

(5 marks)

a) Probabilities add up to 1

$$0.1 + a + 0.25 + b + 0.15 = 1$$

$$a + b = 0.5$$

Use $P(X \leq 1) = P(X \geq 3)$

$$0.1 + a = b + 0.15$$

$$a - b = 0.05$$

Add the two equations

$$2a = 0.55$$

$$a = 0.275$$

Substitute into $a + b = 0.5$

$$b = 0.5 - 0.275$$

$$b = 0.225$$

$$a = 0.275, b = 0.225$$

b) $1 \leq X < 4$ means $X = 1, 2$ or 3

$$P(1 \leq X < 4) = 0.275 + 0.25 + 0.225$$

$$P(1 \leq X < 4) = 0.75$$

$$0.75$$

5

The discrete random variable X takes the values 1, 2, 3, 4 and 5 only. Its cumulative distribution function $F(x) = P(X \leq x)$ is given in the table.

x	1	2	3	4	5
$F(x)$	0.2	0.45	0.7	0.9	1

- Find $P(X = 3)$.
- Find the probability distribution of X .
- Find $P(X > 2)$.

(5 marks)

a) $F(3) - F(2)$ leaves only $X = 3$

$$P(X = 3) = F(3) - F(2)$$

$$P(X = 3) = 0.7 - 0.45$$

$$P(X = 3) = 0.25$$

0.25

b) Each probability is a jump in F

$$P(X = 1) = F(1) = 0.2$$

$$P(X = 2) = 0.45 - 0.2 = 0.25$$

$$P(X = 3) = 0.7 - 0.45 = 0.25$$

$$P(X = 4) = 0.9 - 0.7 = 0.2$$

$$P(X = 5) = 1 - 0.9 = 0.1$$

0.2, 0.25, 0.25, 0.2, 0.1

c) Use the complement

$$P(X > 2) = 1 - P(X \leq 2)$$

$$P(X > 2) = 1 - F(2)$$

$$P(X > 2) = 1 - 0.45 = 0.55$$

0.55

6

The discrete random variable X has probability function

$$P(X = x) = k(x^2 + 1) \quad \text{for } x = 0, 1, 2, 3$$

a. Show that $k = \frac{1}{18}$

b. Find $P(X \geq 2)$.

c. Two independent observations X_1 and X_2 are taken of X . Find $P(X_1 + X_2 = 3)$.

(6 marks)

a) Find each probability

$$P(X = 0) = k(0 + 1) = k$$

$$P(X = 1) = k(1 + 1) = 2k$$

$$P(X = 2) = k(4 + 1) = 5k$$

$$P(X = 3) = k(9 + 1) = 10k$$

Probabilities add up to 1

$$k + 2k + 5k + 10k = 1$$

$$18k = 1$$

$$k = \frac{1}{18}$$

$$k = \frac{1}{18} \text{ as required}$$

b) $X \geq 2$ means $X = 2$ or 3

$$P(X \geq 2) = 5k + 10k$$

$$P(X \geq 2) = \frac{15}{18}$$

$$P(X \geq 2) = \frac{5}{6}$$

$$\frac{5}{6}$$

c) List the pairs that add to 3

$$(0, 3) (3, 0) (1, 2) (2, 1)$$

Independent, so multiply

$$P(0, 3) = \frac{1}{18} \times \frac{10}{18} = \frac{10}{324}$$

$$P(1, 2) = \frac{2}{18} \times \frac{5}{18} = \frac{10}{324}$$

Each pair can happen either way round

$$P(X_1 + X_2 = 3) = 2 \times \frac{10}{324} + 2 \times \frac{10}{324}$$

$$P(X_1 + X_2 = 3) = \frac{40}{324}$$

$$P(X_1 + X_2 = 3) = \frac{10}{81}$$

$$\boxed{\frac{10}{81}}$$

7

A bag contains 3 red counters and 2 blue counters. Ravi takes two counters at random from the bag without replacement. The random variable R is the number of red counters he takes.

- Find the probability distribution of R .
- Find $P(R \leq 1)$.

(5 marks)

a) R can be 0, 1 or 2

$R = 0$: blue then blue

$$P(R = 0) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$$

$R = 2$: red then red

$$P(R = 2) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$$

$R = 1$: red then blue, or blue then red

$$P(R = 1) = \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4}$$

$$P(R = 1) = \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$$

Check: $\frac{1}{10} + \frac{6}{10} + \frac{3}{10} = 1$

$$\boxed{P(R = 0) = \frac{1}{10}, P(R = 1) = \frac{3}{5}, P(R = 2) = \frac{3}{10}}$$

b) $R \leq 1$ means $R = 0$ or 1

$$P(R \leq 1) = \frac{1}{10} + \frac{6}{10}$$

$$P(R \leq 1) = \frac{7}{10}$$

$$\boxed{\frac{7}{10}}$$

8

Two fair four-sided dice, each numbered 1 to 4, are rolled. The random variable S is the larger of the two scores (if the scores are equal, S is that score).

a. Explain why $P(S \leq s) = \frac{s^2}{16}$ for $s = 1, 2, 3, 4$

b. Hence show that $P(S = s) = \frac{2s-1}{16}$ for $s = 1, 2, 3, 4$

c. Find $P(S \geq 3)$.

(6 marks)

a) The larger score is at most s only if both scores are at most s

$$P(\text{one die} \leq s) = \frac{s}{4}$$

The dice are independent, so multiply

$$P(S \leq s) = \frac{s}{4} \times \frac{s}{4}$$

$$P(S \leq s) = \frac{s^2}{16}$$

$$P(S \leq s) = \frac{s^2}{16}$$

b) Use $P(S = s) = P(S \leq s) - P(S \leq s-1)$

$$P(S = s) = \frac{s^2}{16} - \frac{(s-1)^2}{16}$$

$$P(S = s) = \frac{s^2 - (s^2 - 2s + 1)}{16}$$

$$P(S = s) = \frac{2s-1}{16}$$

$$P(S = s) = \frac{2s-1}{16} \text{ as required}$$

c) $S \geq 3$ means $S = 3$ or 4

$$P(S = 3) = \frac{2 \times 3 - 1}{16} = \frac{5}{16}$$

$$P(S = 4) = \frac{2 \times 4 - 1}{16} = \frac{7}{16}$$

$$P(S \geq 3) = \frac{5}{16} + \frac{7}{16}$$

$$P(S \geq 3) = \frac{12}{16} = \frac{3}{4}$$

$$\frac{3}{4}$$

9

The discrete random variable X has probability function

$$P(X = x) = kx \quad \text{for } x = 1, 2, 3$$

$$P(X = x) = k(7 - x) \quad \text{for } x = 4, 5, 6$$

where k is a constant.

- Find the value of k .
- Find $P(2 < X \leq 5)$.
- Find the probability that X is even.

(5 marks)

a) Use the right rule for each value

$$x = 1, 2, 3, 4, 5, 6$$

$$\text{probabilities } k, 2k, 3k, 3k, 2k, k$$

Probabilities add up to 1

$$k + 2k + 3k + 3k + 2k + k = 1$$

$$12k = 1$$

$$k = \frac{1}{12}$$

$$\boxed{k = \frac{1}{12}}$$

b) $2 < X \leq 5$ means $X = 3, 4$ or 5

$$P(2 < X \leq 5) = 3k + 3k + 2k$$

$$P(2 < X \leq 5) = 8k = \frac{8}{12}$$

$$P(2 < X \leq 5) = \frac{2}{3}$$

$$\boxed{\frac{2}{3}}$$

c) Even values: 2, 4, 6

$$P(X \text{ even}) = 2k + 3k + k$$

$$P(X \text{ even}) = 6k = \frac{6}{12}$$

$$P(X \text{ even}) = \frac{1}{2}$$

$$\boxed{\frac{1}{2}}$$

In a game, Megan rolls a fair six-sided die until she gets a six, but she stops after 3 rolls even if she has not got a six. The random variable X is the number of times she rolls the die.

- Find the probability distribution of X .
- Find $P(X \geq 2)$.
- Megan and Tomasz each play the game once, independently. Find the probability that they roll the die the same number of times.

(6 marks)

a) $X = 1$: six on the first roll

$$P(X = 1) = \frac{1}{6}$$

$X = 2$: not a six, then a six

$$P(X = 2) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

$X = 3$: two non-sixes, then she stops whatever happens

$$P(X = 3) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

$$\text{Check: } \frac{1}{6} + \frac{5}{36} + \frac{25}{36} = 1$$

$$P(X = 1) = \frac{1}{6}, P(X = 2) = \frac{5}{36}, P(X = 3) = \frac{25}{36}$$

b) Use the complement

$$P(X \geq 2) = 1 - P(X = 1)$$

$$P(X \geq 2) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\frac{5}{6}$$

c) Both 1, both 2 or both 3 (independent, so multiply)

$$P(\text{same}) = \left(\frac{1}{6}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{25}{36}\right)^2$$

$$P(\text{same}) = \frac{36}{1296} + \frac{25}{1296} + \frac{625}{1296}$$

$$P(\text{same}) = \frac{686}{1296}$$

$$P(\text{same}) = \frac{343}{648}$$

$$\frac{343}{648} \text{ (0.529 to 3 s.f.)}$$

11

The random variable X has a discrete uniform distribution over the integers $1, 2, 3, \dots, n$.

Given that $P(X \leq 6) = 0.4$

- find the value of n
- find $P(4 < X < 11)$
- find the probability that two independent values of X are both greater than 12.

(6 marks)

a) Each value has probability $\frac{1}{n}$

$$P(X \leq 6) = \frac{6}{n}$$

$$\frac{6}{n} = 0.4$$

$$n = \frac{6}{0.4}$$

$$n = 15$$

$$\boxed{n = 15}$$

b) $4 < X < 11$ means $X = 5, 6, 7, 8, 9, 10$

$$P(4 < X < 11) = 6 \times \frac{1}{15}$$

$$P(4 < X < 11) = \frac{6}{15} = \frac{2}{5}$$

$$\boxed{\frac{2}{5}}$$

c) $X > 12$ means $X = 13, 14, 15$

$$P(X > 12) = \frac{3}{15} = \frac{1}{5}$$

Independent, so multiply

$$P(\text{both} > 12) = \frac{1}{5} \times \frac{1}{5}$$

$$P(\text{both} > 12) = \frac{1}{25}$$

$$\boxed{\frac{1}{25}}$$

- 12 The discrete random variable X has probability function

$$P(X = x) = \frac{k}{x} \quad \text{for } x = 1, 2, 3, 6$$

- Show that $k = \frac{1}{2}$
- Find $P(X > 1)$.
- Two independent observations X_1 and X_2 are taken of X . Find $P(X_1 X_2 = 6)$.
- A single value g of X is taken and used in the equation $t^2 + gt + 4 = 0$. Find the probability that this equation has real roots.

(8 marks)

a) Probabilities add up to 1

$$\frac{k}{1} + \frac{k}{2} + \frac{k}{3} + \frac{k}{6} = 1$$

Common denominator 6

$$\frac{6k + 3k + 2k + k}{6} = 1$$

$$\frac{12k}{6} = 1$$

$$2k = 1$$

$$k = \frac{1}{2}$$

$k = \frac{1}{2} \text{ as required}$

b) Write out the distribution

$$P(X = 1) = \frac{1}{2}$$

$$P(X = 2) = \frac{1}{4}$$

$$P(X = 3) = \frac{1}{6}$$

$$P(X = 6) = \frac{1}{12}$$

$$P(X > 1) = 1 - P(X = 1)$$

$$P(X > 1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$\frac{1}{2}$

c) Pairs with product 6

$$(1, 6) (6, 1) (2, 3) (3, 2)$$

$$P(1, 6) = \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}$$

$$P(2, 3) = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$$

Each pair can happen either way round

$$P(X_1 X_2 = 6) = 2 \times \frac{1}{24} + 2 \times \frac{1}{24}$$

$$P(X_1 X_2 = 6) = \frac{4}{24} = \frac{1}{6}$$

$$\boxed{\frac{1}{6}}$$

d) Real roots: discriminant ≥ 0

$$b^2 - 4ac \geq 0$$

$$g^2 - 4 \times 1 \times 4 \geq 0$$

$$g^2 \geq 16$$

g is positive, so $g \geq 4$

Only $g = 6$ works

$$P(\text{real roots}) = P(X = 6) = \frac{1}{12}$$

$$\boxed{\frac{1}{12}}$$