Argand diagrams, modulus and argument

Worked solutions

Further Maths topic practice

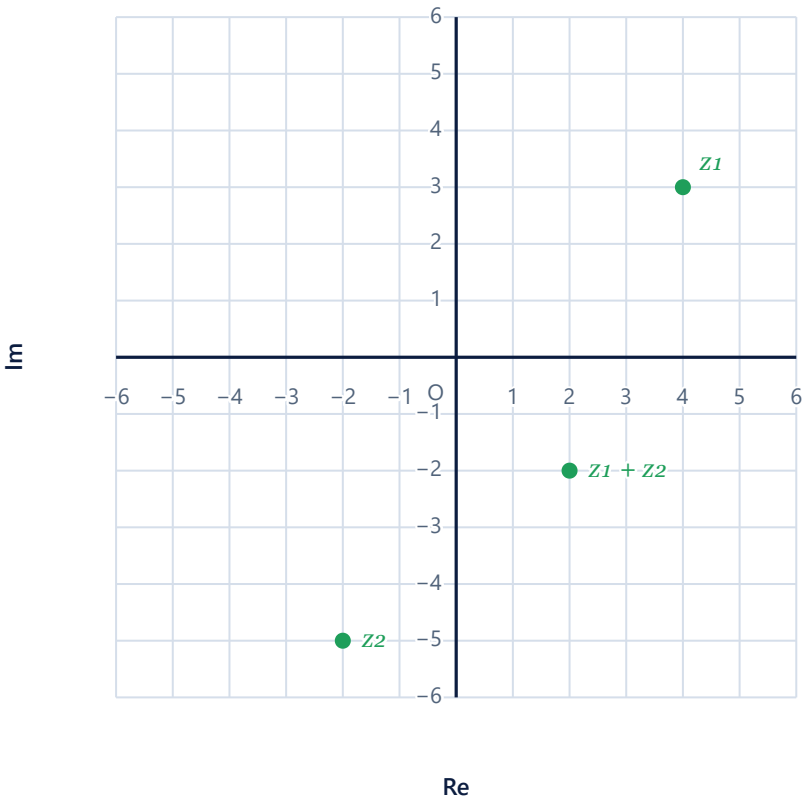
This topic online

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1 The complex numbers z_1 and z_2 are given by $z_1 = 4 + 3i$ and $z_2 = -2 - 5i$

a. Show z_1, z_2 and $z_1 + z_2$ on the Argand diagram.

b. Find the exact value of $|z_1 + z_2|$



(4 marks)

a)

a + bi is plotted at the point (a, b)

$z1 = 4 + 3i$ at $(4, 3)$

$z2 = -2 - 5i$ at $(-2, -5)$

$z1 + z2 = (4 - 2) + (3 - 5)i$

$z1 + z2 = 2 - 2i$

Plot $(4, 3)$, $(-2, -5)$ and $(2, -2)$

b)

Modulus = distance from the origin

$|z1 + z2| = \sqrt{(2^2 + (-2)^2)}$

$|z1 + z2| = \sqrt{8}$

$|z1 + z2| = 2\sqrt{2}$

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2 Find the modulus and the argument of each of the following complex numbers. Give each argument in radians to 3 significant figures, in the range $-\pi < \theta \leq \pi$

a. $z = -3 + 4i$

b. $w = -1 - 2i$

(4 marks)

a)

$$|z| = \sqrt{((-3)^2 + 4^2)}$$

$$|z| = \sqrt{25} = 5$$

Sketch: z is in the second quadrant

Angle with the negative real axis

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 0.9273$$

$$\arg z = \pi - 0.9273$$

$$\arg z = 2.2143$$

$$\boxed{|z| = 5, \arg z = 2.21}$$

b)

$$|w| = \sqrt{((-1)^2 + (-2)^2)}$$

$$|w| = \sqrt{5}$$

Sketch: w is in the third quadrant, below the real axis

$$\alpha = \tan^{-1}\left(\frac{2}{1}\right) = 1.1071$$

$$\arg w = -(\pi - 1.1071)$$

$$\arg w = -2.0344$$

$$\boxed{|w| = \sqrt{5}, \arg w = -2.03}$$

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3 Express each complex number in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$, giving r and θ as exact values.

a. $-\sqrt{3} + i$

b. $2 - 2i$

(4 marks)

a)

$$r = \sqrt{((-\sqrt{3})^2 + 1^2)}$$

$$r = \sqrt{4} = 2$$

Second quadrant

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

b)

$$r = \sqrt{(2^2 + (-2)^2)}$$

$$r = \sqrt{8} = 2\sqrt{2}$$

Fourth quadrant, below the real axis

$$\tan\alpha = \frac{2}{2} = 1$$

$$\alpha = \frac{\pi}{4}$$

$$\theta = -\frac{\pi}{4}$$

$$2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

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4 The complex number $z = 6(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})$

a. Express z in the form $a + bi$, where a and b are exact real numbers.

b. Write down z^* in the form $r(\cos\theta + i \sin\theta)$, where $-\pi < \theta \leq \pi$

c. Find, in the form $a + bi$, the complex number $z + z^*$ and explain why it is real.

(4 marks)

a)

$$\cos \frac{2\pi}{3} = -\frac{1}{2}$$

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$z = 6 \times \left(-\frac{1}{2}\right) + 6 \times \frac{\sqrt{3}}{2} i$$

$$\boxed{z = -3 + 3\sqrt{3}\,i}$$

b)

z^* is the reflection of z in the real axis

Same modulus, argument changes sign

$$\boxed{z^* = 6\left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right)\right)}$$

c)

$$z^* = -3 - 3\sqrt{3}\,i$$

$$z + z^* = (-3 - 3) + (3\sqrt{3} - 3\sqrt{3})i$$

$$z + z^* = -6$$

$$\boxed{-6\text{: the imaginary parts cancel, } z + z^* = 2\,\mathrm{Re}(z)}$$

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5 The complex number z is given by $z = \dfrac{5 + 5i}{1 + 2i}$

a. Express z in the form $a + bi$, where a and b are real.

b. Find the exact value of $|z|$ and find arg z in radians to 3 significant figures.

c. Hence find $|z^2|$ and arg(z^2), giving the argument in radians to 3 significant figures. (6 marks)

a)

Multiply top and bottom by $1 - 2i$

$$z = \frac{(5 + 5i)(1 - 2i)}{((1 + 2i)(1 - 2i))}$$

$$z = \frac{(5 - 10i + 5i - 10i^2)}{5}$$

$$z = \frac{(15 - 5i)}{5}$$

$$\boxed{z = 3 - i}$$

b)

$$|z| = \sqrt{(3^2 + (-1)^2)} = \sqrt{10}$$

Fourth quadrant

$$\arg z = -\tan^{-1}\left(\frac{1}{3}\right)$$

$$\arg z = -0.32175$$

$$\boxed{|z| = \sqrt{10},\ \arg z = -0.322}$$

c)

Multiplying: module multiply, arguments add

$$|z^2| = \sqrt{10} \times \sqrt{10} = 10$$

$$\arg(z^2) = 2 \times (-0.32175)$$

$$\arg(z^2) = -0.6435$$

Check: $z^2 = 9 - 6i - 1 = 8 - 6i$, in the fourth quadrant

$$\boxed{|z^2| = 10,\ \arg(z^2) = -0.644}$$

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6 $z_1 = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ and $z_2 = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

Find, in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$,

a. $z_1 z_2$

b. $\frac{z_1}{z_2}$

(5 marks)

a)

Product: multiply moduli, add arguments

$$|z_1 z_2| = 4 \times 2 = 8$$

$$\arg(z_1 z_2) = \frac{\pi}{3} + \frac{3\pi}{4}$$

$$\arg(z_1 z_2) = \frac{4\pi}{12} + \frac{9\pi}{12} = \frac{13\pi}{12}$$

Outside $-\pi < \theta \leq \pi$, so subtract 2π

$$\theta = \frac{13\pi}{12} - 2\pi = -\frac{11\pi}{12}$$

$$8\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$$

b)

Quotient: divide moduli, subtract arguments

$$\left|\frac{z_1}{z_2}\right| = \frac{4}{2} = 2$$

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{3} - \frac{3\pi}{4}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{4\pi}{12} - \frac{9\pi}{12} = -\frac{5\pi}{12}$$

$$2\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)$$

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7 The complex number $z = p + 2i$, where p is a real constant, has argument $\frac{2\pi}{3}$

a. Find the exact value of p

b. Find the exact value of $|z|$

The complex number $w = 1 - i$

c. Find the exact values of $|zw|$ and $\arg(zw)$.

(6 marks)

a)

$\arg z = \frac{2\pi}{3}$: second quadrant, so $p < 0$

Angle with the negative real axis is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$$\tan\frac{\pi}{3} = \frac{2}{(-p)}$$

$$\sqrt{3} = \frac{2}{(-p)}$$

$$-p = \frac{2}{\sqrt{3}}$$

$$p = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\boxed{p = -\frac{2\sqrt{3}}{3}}$$

b)

$$|z|^2 = \left(-\frac{2\sqrt{3}}{3}\right)^2 + 2^2$$

$$|z|^2 = \frac{12}{9} + 4 = \frac{16}{3}$$

$$|z| = \frac{4}{\sqrt{3}}$$

$$\boxed{|z| = \frac{4\sqrt{3}}{3}}$$

c)

$$|w| = \sqrt{(1+1)} = \sqrt{2}$$

w is in the fourth quadrant

$$\arg w = -\frac{\pi}{4}$$

Modulus multiply, arguments add

$$|zw| = \frac{4\sqrt{3}}{3} \times \sqrt{2} = \frac{4\sqrt{6}}{3}$$

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$$\arg(zw) = \frac{2\pi}{3} - \frac{\pi}{4}$$

$$\arg(zw) = \frac{8\pi}{12} - \frac{3\pi}{12} = \frac{5\pi}{12}$$

$$\boxed{|zw| = \frac{4\sqrt{6}}{3}, \arg(zw) = \frac{5\pi}{12}}$$

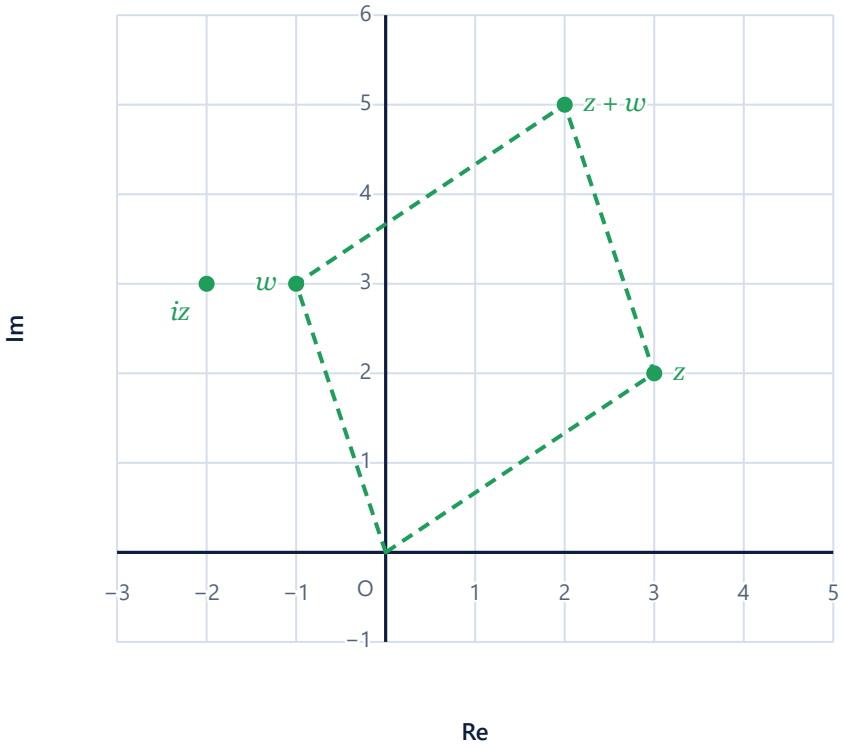
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8 The complex numbers z and w are given by $z = 3 + 2i$ and $w = -1 + 3i$

a. On the Argand diagram, show the points representing z, w and $z + w$

b. Name the quadrilateral whose vertices are the origin and the points representing z, $z + w$ and w, giving a reason.

c. Find iz in the form $a + bi$ and describe fully the single transformation that maps the point representing z to the point representing iz



(6 marks)

a)

$z + w = (3 - 1) + (2 + 3)i$

$z + w = 2 + 5i$

Plot (3, 2), (-1, 3) and (2, 5)

b)

Addition of complex numbers is vector addition

From O to w is the vector w

From z to z + w is also the vector w

Opposite sides equal and parallel

A parallelogram

c)

$iz = i(3 + 2i)$

$iz = 3i + 2i^2$

$iz = -2 + 3i$

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$|i| = 1$ and $\arg i = \frac{\pi}{2}$

So $|iz| = |z|$ and $\arg(iz) = \arg z + \frac{\pi}{2}$

$iz = -2 + 3i$; rotation 90° anticlockwise about O

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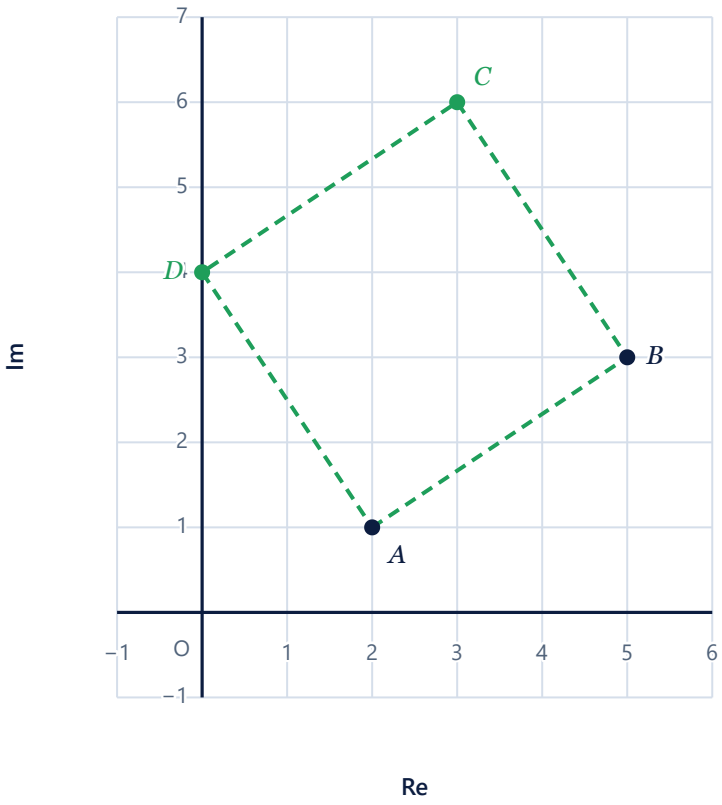
9 In an Argand diagram, the points A and B represent the complex numbers $2 + i$ and $5 + 3i$ respectively.

The points A, B, C and D are the vertices of a square, labelled anticlockwise.

a. Write down the complex number that represents the vector $\vec{AB}$

b. By multiplying by i, find the complex numbers represented by C and D

c. Find the area of the square.



(6 marks)

a)

$AB = (5 + 3i) - (2 + i)$

$\boxed{3 + 2i}$

b)

Anticlockwise square: BC is AB rotated 90° anticlockwise

Multiplying by i rotates 90° anticlockwise

$BC = i(3 + 2i)$

$BC = 3i + 2i^2$

$BC = -2 + 3i$

$C = (5 + 3i) + (-2 + 3i)$

$C = 3 + 6i$

AD is the same vector as BC

$D = (2 + i) + (-2 + 3i)$

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$O = 4i$

$\boxed{C: 3 + 6i,\ O: 4i}$

c)

side $= |3 + 2i| = \sqrt{(9 + 4)} = \sqrt{13}$

area $= (\sqrt{13})^2$

$\boxed{\text{area} = 13}$

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10 The point A on an Argand diagram represents the complex number $a = 2\sqrt{3} + 2i$

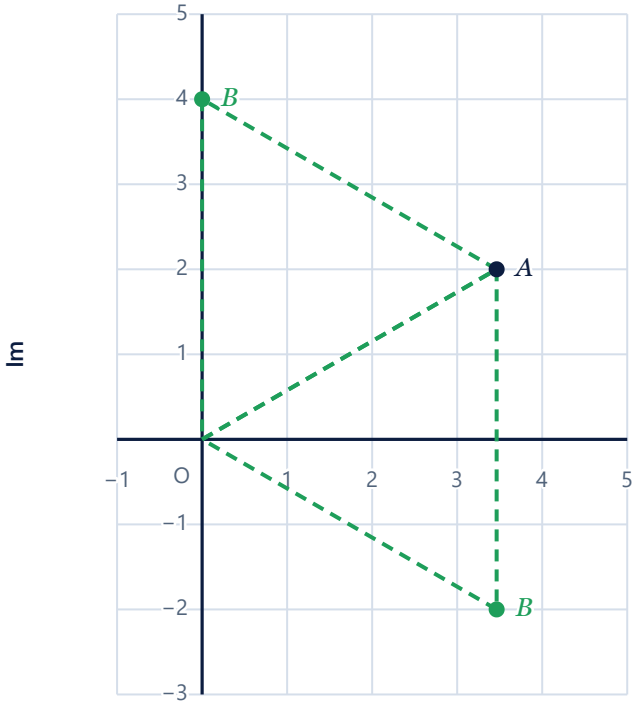
a. Express a in the form $r(\cos\theta + i\sin\theta)$, giving r and θ exactly.

The point B represents the complex number b. The origin O, A and B are the vertices of an equilateral triangle.

b. Explain why $b = a(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ or $b = a(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$.

c. Find the two possible values of b, giving each in the form $x + yi$

d. Find the exact area of triangle OAB



(7 marks)

a)

$$r = \sqrt{((2\sqrt{3})^2 + 2^2)}$$

$$r = \sqrt{16} = 4$$

$$\tan\theta = \frac{2}{(2\sqrt{3})} = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$$a = 4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$

b)

Equilateral: OB = OA and angle AOB = $\frac{\pi}{3}$

$\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}$ has modulus 1 and argument $\pm\frac{\pi}{3}$

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Multiplying by it rotates a by $\frac{\pi}{3}$ about O, keeping $|b| = |a|$

B is A rotated $\pm 60^\circ$ about O

c)

Modulus multiply, arguments add

$$b = 4\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right)$$

$$b = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$b = 4(0 + i)$$

$$b = 4i$$

Other way

$$b = 4\left(\cos\left(\frac{\pi}{6} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\right)$$

$$b = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$b = 4\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$b = 2\sqrt{3} - 2i$$

$b = 4i$ or $b = 2\sqrt{3} - 2i$

d)

Area $= \frac{1}{2}ab\sin C$

$$\text{area} = \frac{1}{2} \times 4 \times 4 \times \sin\frac{\pi}{3}$$

$$\text{area} = 8 \times \frac{\sqrt{3}}{2}$$

$\text{area} = 4\sqrt{3}$

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11 The complex numbers $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$

a. Write down the modulus and argument of z_1 and of z_2, giving exact values.

b. Hence find $|z_1 z_2|$, $\arg(z_1 z_2)$, $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$.

c. Express $z_1 z_2$ in the form $a + bi$

d. Hence show that $\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

(8 marks)

a)

$$|z1| = \sqrt{(1+1)} = \sqrt{2}$$

$$\arg z1 = \frac{\pi}{4}$$

$$|z2| = \sqrt{(3+1)} = 2$$

z2 is in the fourth quadrant, $\tan\alpha = \frac{1}{\sqrt{3}}$

$$\arg z2 = -\frac{\pi}{6}$$

$$\boxed{\sqrt{2}, \frac{\pi}{4} \text{ and } 2, -\frac{\pi}{6}}$$

b)

Product: multiply moduli, add arguments

$$|z1z2| = \sqrt{2} \times 2 = 2\sqrt{2}$$

$$\arg(z1z2) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

Quotient: divide moduli, subtract arguments

$$\left|\frac{z1}{z2}\right| = \frac{\sqrt{2}}{2}$$

$$\arg\left(\frac{z1}{z2}\right) = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$$

$$\boxed{2\sqrt{2}, \frac{\pi}{12}; \frac{\sqrt{2}}{2}, \frac{5\pi}{12}}$$

c)

$$z1z2 = (1+i)(\sqrt{3}-i)$$

$$z1z2 = \sqrt{3} - i + \sqrt{3}i - i^2$$

$$\boxed{z1z2 = (\sqrt{3}+1) + (\sqrt{3}-1)i}$$

d)

Real part = modulus × cos(argument)

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$$2\sqrt{2}\cos\frac{\pi}{12} = \sqrt{3}+1$$

$$\cos\frac{\pi}{12} = \frac{(\sqrt{3}+1)}{(2\sqrt{2})}$$

Multiply top and bottom by $\sqrt{2}$

$$\cos\frac{\pi}{12} = \frac{(\sqrt{6}+\sqrt{2})}{4}$$

$$\boxed{\cos\frac{\pi}{12} = \frac{(\sqrt{6}+\sqrt{2})}{4}}$$

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12 The complex number z is given by

$$z = \frac{a + 2i}{1 + i}$$

where a is a real constant.

a. Show that $z = \frac{a + 2}{2} + \frac{2 - a}{2} i$

Given that $\arg z = \frac{\pi}{6}$

b. show that $a = 4 - 2\sqrt{3}$

c. find the exact value of $|z|$, giving your answer in the form $p\sqrt{3} + q$, where p and q are integers. (8 marks)

a)

Multiply top and bottom by $1 - i$

$$z = \frac{(a + 2i)(1 - i)}{((1 + i)(1 - i))}$$

$$z = \frac{(a - ai + 2i - 2i^2)}{2}$$

$$z = \frac{((a + 2) + (2 - a)i)}{2}$$

$$\boxed{z = \frac{(a + 2)}{2} + \frac{(2 - a)}{2} i}$$

b)

$\arg z = \frac{\pi}{6}$: first quadrant, $\text{Im} \div \text{Re} = \tan \frac{\pi}{6}$

$$\frac{(2 - a)}{(a + 2)} = \frac{1}{\sqrt{3}}$$

$$\sqrt{3}(2 - a) = a + 2$$

$$2\sqrt{3} - \sqrt{3}a = a + 2$$

$$2\sqrt{3} - 2 = a + \sqrt{3}a$$

$$a = \frac{(2\sqrt{3} - 2)}{(1 + \sqrt{3})}$$

Rationalise: multiply top and bottom by $\sqrt{3} - 1$

$$a = \frac{(2\sqrt{3} - 2)(\sqrt{3} - 1)}{((\sqrt{3} + 1)(\sqrt{3} - 1))}$$

$$a = \frac{(6 - 2\sqrt{3} - 2\sqrt{3} + 2)}{2}$$

$$a = \frac{(8 - 4\sqrt{3})}{2}$$

$$\boxed{a = 4 - 2\sqrt{3}}$$

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c)

$$\text{Re } z = \frac{(a+2)}{2} = \frac{(6-2\sqrt{3})}{2} = 3-\sqrt{3}$$

$$\text{Re } z = |z| \cos\frac{\pi}{6}$$

$$|z| = \frac{(3-\sqrt{3})}{(\sqrt{3}/2)}$$

$$|z| = \frac{2(3-\sqrt{3})}{\sqrt{3}}$$

$$|z| = \frac{(6-2\sqrt{3})}{\sqrt{3}}$$

Multiply top and bottom by $\sqrt{3}$

$$|z| = \frac{(6\sqrt{3}-6)}{3}$$

$$\boxed{|z| = 2\sqrt{3}-2}$$

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