Complex numbers: arithmetic and quadratics

Worked solutions

Further Maths topic practice

This topic online

1 The complex numbers z and w are given by $z = 3 - 4i$ and $w = -2 + 5i$

Find, in the form $a + bi$, where a and b are real,

a. $z + 2w$

b. zw

c. z^2

(4 marks)

a)

Add real parts and imaginary parts separately

$z + 2w = (3 - 4i) + 2(-2 + 5i)$

$z + 2w = 3 - 4i - 4 + 10i$

$\boxed{z + 2w = -1 + 6i}$

b)

Expand the brackets, then use $i^2 = -1$

$zw = (3 - 4i)(-2 + 5i)$

$zw = -6 + 15i + 8i - 20i^2$

$zw = -6 + 23i + 20$

$\boxed{zw = 14 + 23i}$

c)

$z^2 = (3 - 4i)(3 - 4i)$

$z^2 = 9 - 12i - 12i + 16i^2$

$z^2 = 9 - 24i - 16$

$\boxed{z^2 = -7 - 24i}$

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2 Express $\frac{5+i}{2-3i}$ in the form $a + bi$, where a and b are real. (3 marks)

Multiply top and bottom by the conjugate $2 + 3i$

$$\frac{(5+i)}{(2-3i)} = \frac{(5+i)(2+3i)}{((2-3i)(2+3i))}$$

Numerator

$$(5+i)(2+3i) = 10 + 15i + 2i + 3i^2$$

$$(5+i)(2+3i) = 10 + 17i - 3$$

$$(5+i)(2+3i) = 7 + 17i$$

Denominator

$$(2-3i)(2+3i) = 4 - 9i^2$$

$$(2-3i)(2+3i) = 4 + 9 = 13$$

$$\boxed{\frac{7}{13} + \frac{17}{13}i}$$

3 Solve the equation $2z^2 + 6z + 5 = 0$, giving your answers in the form $p + qi$, where p and q are real. (3 marks)

Quadratic formula with $a = 2$, $b = 6$, $c = 5$

$$z = \frac{(-6 \pm \sqrt{(6^2 - 4 \times 2 \times 5)})}{(2 \times 2)}$$

$$z = \frac{(-6 \pm \sqrt{(36 - 40)})}{4}$$

$$z = \frac{(-6 \pm \sqrt{(-4)})}{4}$$

$\sqrt{(-4)} = \sqrt{4} \times \sqrt{(-1)} = 2i$

$$z = \frac{(-6 \pm 2i)}{4}$$

$$\boxed{z = -\frac{3}{2} + \frac{1}{2}i \text{ or } z = -\frac{3}{2} - \frac{1}{2}i}$$

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4 Find the complex number z that satisfies the equation

$$3z - 2iz^* = 14 - 11i$$

Give your answer in the form $a + bi$, where a and b are real. (5 marks)

Let $z = a + bi$, so $z^* = a - bi$

$$3z = 3a + 3bi$$

$$2iz^* = 2i(a - bi)$$

$$2iz^* = 2ai - 2bi^2$$

$$2iz^* = 2b + 2ai$$

$$3z - 2iz^* = (3a - 2b) + (3b - 2a)i$$

Equate real parts

$$3a - 2b = 14$$

Equate imaginary parts

$$-2a + 3b = -11$$

Solve simultaneously: × 3 and × 2, then add

$$9a - 6b = 42$$

$$-4a + 6b = -22$$

$$5a = 20$$

$$a = 4$$

$$3(4) - 2b = 14$$

$$b = -1$$

$$\boxed{z = 4 - i}$$

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5 The equation $z^2 + pz + q = 0$, where p and q are real constants, has a root $2 - i$

a. Write down the other root of the equation.

b. Find the value of p and the value of q

(4 marks)

a)

Real coefficients, so roots come in conjugate pairs

$\boxed{2 + i}$

b)

$z^2 + pz + q = (z - \alpha)(z - \beta)$, so sum $= -p$ and product $= q$

$$\alpha + \beta = (2 - i) + (2 + i) = 4$$

$$-p = 4$$

$$p = -4$$

$$\alpha\beta = (2 - i)(2 + i)$$

$$\alpha\beta = 4 - i^2$$

$$\alpha\beta = 4 + 1 = 5$$

$\boxed{p = -4, q = 5}$

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6 The complex number w is given by

$$w = \frac{a + 4i}{1 - 2i}$$

where a is a real constant.

a. Express w in the form $x + yi$, where x and y are given in terms of a

b. Given that w is purely imaginary, find the value of a and the value of w

c. Find the value of a for which w is real.

(5 marks)

a)

Multiply top and bottom by the conjugate $1 + 2i$

$$w = \frac{(a + 4i)(1 + 2i)}{((1 - 2i)(1 + 2i))}$$

$$(a + 4i)(1 + 2i) = a + 2ai + 4i + 8i^2$$

$$(a + 4i)(1 + 2i) = (a - 8) + (2a + 4)i$$

$$(1 - 2i)(1 + 2i) = 1 + 4 = 5$$

$$\boxed{w = \frac{(a - 8)}{5} + \frac{(2a + 4)}{5}i}$$

b)

Purely imaginary: real part = 0

$$\frac{(a - 8)}{5} = 0$$

$$a = 8$$

$$w = \frac{(2 \times 8 + 4)}{5}i$$

$$\boxed{a = 8, w = 4i}$$

c)

Real: imaginary part = 0

$$\frac{(2a + 4)}{5} = 0$$

$$\boxed{a = -2}$$

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7 Given that $z = 2 + i$

a. find $\frac{1}{z}$ in the form $a + bi$, where a and b are real.

The complex number w is defined by $w = z + \frac{k}{z}$, where k is a real constant.

b. Given that w is real, find the value of k and the value of w

(5 marks)

a)

Multiply top and bottom by $z^* = 2 - i$

$$\frac{1}{z} = \frac{(2 - i)}{((2 + i)(2 - i))}$$

$$\frac{1}{z} = \frac{(2 - i)}{(4 + 1)}$$

$$\boxed{\frac{1}{z} = \frac{2}{5} - \frac{1}{5}i}$$

b)

$$w = 2 + i + k\left(\frac{2}{5} - \frac{1}{5}i\right)$$

Collect real and imaginary parts

$$w = \left(2 + \frac{2k}{5}\right) + \left(1 - \frac{k}{5}\right)i$$

w is real, so the imaginary part is 0

$$1 - \frac{k}{5} = 0$$

$$k = 5$$

$$w = 2 + \frac{2 \times 5}{5}$$

$$\boxed{k = 5, w = 4}$$

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8 Find the two complex numbers z that satisfy the equation

$$zz^* + 2iz = 12 + 4i$$

Give your answers in the form $a + bi$, where a and b are real. (6 marks)

Let $z = a + bi$

$$zz^* = (a + bi)(a - bi) = a^2 + b^2$$

$$2iz = 2ai + 2bi^2$$

$$2iz = -2b + 2ai$$

$$zz^* + 2iz = (a^2 + b^2 - 2b) + 2ai$$

Equate imaginary parts

$$2a = 4$$

$$a = 2$$

Equate real parts

$$a^2 + b^2 - 2b = 12$$

$$4 + b^2 - 2b = 12$$

$$b^2 - 2b - 8 = 0$$

$$(b - 4)(b + 2) = 0$$

$$b = 4 \text{ or } b = -2$$

$$\boxed{z = 2 + 4i \text{ or } z = 2 - 2i}$$

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9 a. Find the two square roots of $21 - 20i$, giving your answers in the form $a + bi$, where a and b are real.

b. Hence solve the equation

$$z^2 - 3z - 3 + 5i = 0$$

(7 marks)

a)

Let $(a + bi)^2 = 21 - 20i$, a and b real

$a^2 - b^2 + 2abi = 21 - 20i$

Equate real and imaginary parts

$a^2 - b^2 = 21$

$2ab = -20$

$b = -\frac{10}{a}$

Substitute into the first equation

$a^2 - \frac{100}{a^2} = 21$

$a^4 - 21a^2 - 100 = 0$

$(a^2 - 25)(a^2 + 4) = 0$

a is real, so $a^2 = 25$

$a = 5, b = -2$ or $a = -5, b = 2$

$\boxed{5 - 2i \text{ and } -5 + 2i}$

b)

Quadratic formula with $a = 1, b = -3, c = -3 + 5i$

$b^2 - 4ac = 9 - 4(-3 + 5i)$

$b^2 - 4ac = 9 + 12 - 20i$

$b^2 - 4ac = 21 - 20i$

Use the square roots from part (a)

$z = \frac{(3 \pm (5 - 2i))}{2}$

$z = \frac{(8 - 2i)}{2}$ or $z = \frac{(-2 + 2i)}{2}$

$\boxed{z = 4 - i \text{ or } z = -1 + i}$

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10 The complex numbers z and w satisfy the simultaneous equations

$$z + 2w = 7$$
$$iz - w = -4 + 4i$$

Find z and w, giving each in the form $a + bi$, where a and b are real. (6 marks)

Make w the subject of the second equation

$$w = iz + 4 - 4i$$

Substitute into the first equation

$$z + 2(iz + 4 - 4i) = 7$$
$$z + 2iz + 8 - 8i = 7$$
$$z(1 + 2i) = -1 + 8i$$
$$z = \frac{(-1 + 8i)}{(1 + 2i)}$$

Multiply top and bottom by $1 - 2i$

$$z = \frac{(-1 + 8i)(1 - 2i)}{((1 + 2i)(1 - 2i))}$$
$$z = \frac{(-1 + 2i + 8i - 16i^2)}{5}$$
$$z = \frac{(15 + 10i)}{5}$$
$$z = 3 + 2i$$

Substitute back for w

$$w = i(3 + 2i) + 4 - 4i$$
$$w = 3i + 2i^2 + 4 - 4i$$
$$w = 3i - 2 + 4 - 4i$$

$$\boxed{z = 3 + 2i,\ w = 2 - i}$$

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11 The quadratic equation

$$x^2 + 2kx + 3k + 4 = 0$$

where k is a real constant, has roots that are not real.

a. Find the range of possible values of k

Given now that $k = 2$

b. solve the equation, giving your answers in the form $p + qi$, where p and q are real and exact,

c. show that the sum of the squares of the two roots is a real number, and find it.

(8 marks)

a)

Not real roots: $b^2 - 4ac < 0$

$(2k)^2 - 4(1)(3k + 4) < 0$

$4k^2 - 12k - 16 < 0$

$k^2 - 3k - 4 < 0$

$(k - 4)(k + 1) < 0$

Critical values -1 and 4; a u-shaped graph is negative between them

$$\boxed{-1 < k < 4}$$

b)

$k = 2$ gives

$$x^2 + 4x + 10 = 0$$

$$x = \frac{(-4 \pm \sqrt{(16 - 40)})}{2}$$

$$x = \frac{(-4 \pm \sqrt{(-24)})}{2}$$

$\sqrt{(-24)} = \sqrt{24}\,i = 2\sqrt{6}\,i$

$$x = \frac{(-4 \pm 2\sqrt{6}\,i)}{2}$$

$$\boxed{x = -2 + \sqrt{6}\,i \text{ or } x = -2 - \sqrt{6}\,i}$$

c)

Use $\alpha + \beta = -4$ and $\alpha\beta = 10$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^2 + \beta^2 = (-4)^2 - 2 \times 10$$

$$\alpha^2 + \beta^2 = 16 - 20$$

Check directly: $(-2 + \sqrt{6}\,i)^2 = 4 - 4\sqrt{6}\,i - 6 = -2 - 4\sqrt{6}\,i$

and $(-2 - \sqrt{6}\,i)^2 = -2 + 4\sqrt{6}\,i$, so the imaginary parts cancel

$$\boxed{\alpha^2 + \beta^2 = -4, \text{ which is real}}$$

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12 a. Find the square roots of $-5 + 12i$, giving your answers in the form $a + bi$, where a and b are real.

b. Hence, showing all your working, find the four roots of the equation

$$z^4 + 10z^2 + 169 = 0$$

c. Use your answer to part (b) to write $z^4 + 10z^2 + 169$ as the product of two quadratic factors with real coefficients.

(9 marks)

a)

Let $(a + bi)^2 = -5 + 12i$

$$a^2 - b^2 = -5$$

$$2ab = 12$$

$$b = \frac{6}{a}$$

$$a^2 - \frac{36}{a^2} = -5$$

$$a^4 + 5a^2 - 36 = 0$$

$$(a^2 + 9)(a^2 - 4) = 0$$

a is real, so $a^2 = 4$

$a = 2, b = 3$ or $a = -2, b = -3$

$\boxed{2 + 3i \text{ and } -2 - 3i}$

b)

Quadratic in z^2

$$z^2 = \frac{(-10 \pm \sqrt{(100 - 4 \times 169)})}{2}$$

$$z^2 = \frac{(-10 \pm \sqrt{(-576)})}{2}$$

$$z^2 = \frac{(-10 \pm 24i)}{2}$$

$z^2 = -5 + 12i$ or $z^2 = -5 - 12i$

From part (a)

$z^2 = -5 + 12i$ gives $z = \pm(2 + 3i)$

$-5 - 12i$ is the conjugate, so its square roots are the conjugates

$z^2 = -5 - 12i$ gives $z = \pm(2 - 3i)$

$\boxed{z = 2 + 3i, 2 - 3i, -2 + 3i, -2 - 3i}$

c)

Pair each root with its conjugate: $2 \pm 3i$ and $-2 \pm 3i$

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Sum 4, product 4 + 9 = 13

$$(z - (2 + 3i))(z - (2 - 3i)) = z^2 - 4z + 13$$

Sum -4, product 13

$$(z - (-2 + 3i))(z - (-2 - 3i)) = z^2 + 4z + 13$$

Check: the product has z^2 coefficient 13 + 13 − 16 = 10

$$\boxed{(z^2 - 4z + 13)(z^2 + 4z + 13)}$$

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