Linear transformations

Worked solutions

Further Maths topic practice

This topic online

1 Write down the 2 × 2 matrix that represents

a. a rotation of 90° anticlockwise about the origin

b. a reflection in the line $y = -x$

c. a stretch parallel to the y-axis, scale factor 4

(3 marks)

Columns are the images of (1; 0) and (0; 1)

Rows separated by ;

a)

(1; 0) goes to (0; 1), (0; 1) goes to (−1; 0)

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b)

(1; 0) goes to (0; −1), (0; 1) goes to (−1; 0)

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

c)

(1; 0) stays, (0; 1) goes to (0; 4)

$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

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2 Describe fully the single geometrical transformation represented by each matrix.

a. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

b. $\begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix}$

c. $\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$

(4 marks)

a)

$(x; y)$ goes to $(y; x)$

Reflection in the line $y = x$

b)

Rotation matrix $(\cos\theta, -\sin\theta; \sin\theta, \cos\theta)$

$\cos\theta = -\frac{1}{2}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

both negative: θ is in the third quadrant

$\theta = 180^\circ + 60^\circ = 240^\circ$

Rotation 240° anticlockwise about O

c)

$(x; y)$ goes to $(-2x; -2y)$

Enlargement, scale factor -2, centre O

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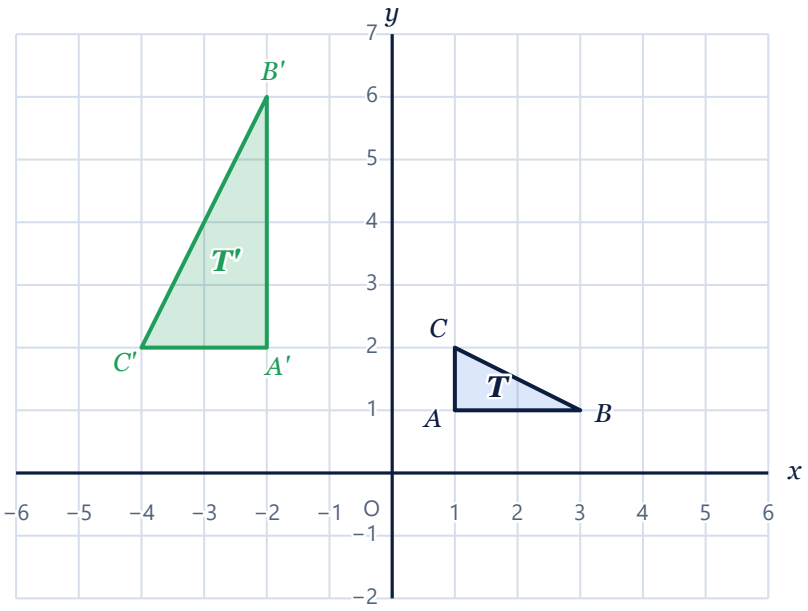
3 The triangle T has vertices $A(1, 1)$, $B(3, 1)$ and $C(1, 2)$, as shown on the grid.

T is transformed to the triangle T' by the matrix $\mathbf{M} = \begin{vmatrix} 0 & -2 \\ 2 & 0 \end{vmatrix}$

a. Find the coordinates of the vertices of T' and draw T' on the grid.

b. Find the area of T' using det **M**

c. Describe the transformation represented by **M** as a combination of two transformations.



(5 marks)

a)

Put the vertices as columns and multiply

Rows separated by ;

$(0, -2; 2, 0)(1, 3, 1; 1, 1, 2) = (-2, -2, -4; 2, 6, 2)$

$\boxed{A'(-2, 2), B'(-2, 6), C'(-4, 2)}$

b)

$\text{area of } T = \frac{1}{2} \times 2 \times 1 = 1$

$\det M = 0 \times 0 - (-2) \times 2 = 4$

area of image = $|\det M|$ x area of object

area of $T' = 4 \times 1$

$\boxed{\text{area of } T' = 4}$

c)

$M = 2(0, -1; 1, 0)$

$\boxed{\text{Rotation } 90^\circ \text{ anticlockwise about O and enlargement, scale factor 2, centre O}}$

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4 The transformation P is a rotation of 90° clockwise about the origin. The transformation Q is a reflection in the line $y = -x$

a. Write down the matrices **P** and **Q**

b. Find the matrix representing the transformation P followed by Q.

c. Describe fully the single transformation equivalent to P followed by Q.

(4 marks)

a)

Rows separated by ;

$P = (0, 1; -1, 0)$

$Q = (0, -1; -1, 0)$

b)

P first, so P goes on the right: QP

$QP = (0, -1; -1, 0)(0, 1; -1, 0)$

$QP = (0 \times 0 + (-1)(-1), 0 \times 1 + (-1) \times 0; -1 \times 0 + 0 \times (-1), -1 \times 1 + 0 \times 0)$

$\boxed{QP = (1, 0; 0, -1)}$

c)

$(x; y)$ goes to $(x; -y)$

$\boxed{\text{Reflection in the } x\text{-axis}}$

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5 The matrix $\mathbf{M} = \begin{vmatrix} k & 3 \\ 2 & k+1 \end{vmatrix}$, where k is a positive constant.

A region R of area 5 cm^2 is transformed by **M** to a region R' of area 70 cm^2

a. Find the value of k

b. Explain how you know that the transformation preserves orientation.

c. Using your value of k, find the point that **M** maps to (13, 17).

(6 marks)

a)

$$\text{area scale factor} = \frac{70}{5} = 14$$

$$\det M = k(k+1) - 3 \times 2 = k^2 + k - 6$$

$$|\det M| = 14$$

$$k^2 + k - 6 = 14$$

$$k^2 + k - 20 = 0$$

$$(k+5)(k-4) = 0$$

$k = 4$ (k is positive)

The other case: $k^2 + k - 6 = -14$ gives $k^2 + k + 8 = 0$, discriminant $1 - 32 < 0$, no real roots

$\boxed{k = 4}$

b)

$$\det M = 16 + 4 - 6 = 14$$

$\boxed{\det M > 0, \text{ so orientation is preserved}}$

c)

Rows separated by ;

$$M = (4, 3; 2, 5)$$

$$M^{-1} = \frac{1}{14}(5, -3; -2, 4)$$

$$M^{-1}(13; 17) = \frac{1}{14}(65 - 51; -26 + 68)$$

$$M^{-1}(13; 17) = \frac{1}{14}(14; 42)$$

$\boxed{(1, 3)}$

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6 The transformation T is represented by the matrix $\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}$

Show that T has a line of invariant points, and find its equation. (4 marks)

Invariant point: $M(x; y) = (x; y)$

Rows separated by ;

$$(3, -2; 1, 0)(x; y) = (x; y)$$

$$3x - 2y = x$$

$$x = y$$

First equation

$$2x = 2y$$

$$y = x$$

The second equation is also $y = x$

Check with (5, 5): $(15 - 10; 5) = (5; 5)$

Every point on $y = x$ is invariant: line of invariant points $y = x$

7 $\mathbf{M} = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$

Find the equations of the lines through the origin, of the form $y = mx$, that are invariant under the transformation represented by **M** (5 marks)

Take a general point on the line: (x, mx)

Rows separated by ;

$$(4, 1; 2, 3)(x; mx) = (4x + mx; 2x + 3mx)$$

Image must lie on $y = mx$

$$2x + 3mx = m(4x + mx)$$

Divide by x

$$2 + 3m = 4m + m^2$$

$$m^2 + m - 2 = 0$$

$$(m + 2)(m - 1) = 0$$

$$m = 1 \text{ or } m = -2$$

$y = x$ and $y = -2x$

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8 The transformation S is a rotation of 45° anticlockwise about the origin followed by an enlargement, scale factor $2\sqrt{2}$, centre the origin.

a. Show that S is represented by the matrix $\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix}$

b. Find the image of the point (3, −1) under S.

c. The point P is mapped by S to the point (2, 10). Find the coordinates of P

(6 marks)

a)

Rows separated by ;

rotation = $(\cos 45^\circ, -\sin 45^\circ; \sin 45^\circ, \cos 45^\circ)$

rotation = $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

enlargement = $(2\sqrt{2}, 0; 0, 2\sqrt{2})$

Rotation first, so the enlargement goes on the left

$S = 2\sqrt{2} \times \frac{1}{\sqrt{2}}(1, -1; 1, 1)$

$\boxed{S = (2, -2; 2, 2) \text{ (shown)}}$

b)

$(2, -2; 2, 2)(3; -1) = (6 + 2; 6 - 2)$

$\boxed{(8, 4)}$

c)

$\det = 2 \times 2 - (-2) \times 2 = 8$

$S^{-1} = \frac{1}{8}(2, 2; -2, 2)$

$P = \frac{1}{8}(2, 2; -2, 2)(2; 10)$

$P = \frac{1}{8}(4 + 20; -4 + 20)$

$P = \frac{1}{8}(24; 16)$

$\boxed{P(3, 2)}$

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9 The transformation U is a rotation of 90° anticlockwise about the x-axis. The transformation V is a reflection in the plane $z = 0$

a. Write down the 3 × 3 matrices **U** and **V**

b. Find the matrix representing U followed by V.

c. Find the image of the point (2, −1, 3) under U followed by V.

d. Find the matrix representing V followed by U, and state whether the order of U and V matters. (6 marks)

a)

Rotation about the x-axis: x is unchanged

Rows separated by ;

$U = (1, 0, 0; 0, \cos 90^\circ, -\sin 90^\circ; 0, \sin 90^\circ, \cos 90^\circ)$

$U = (1, 0, 0; 0, 0, -1; 0, 1, 0)$

Reflection in $z = 0$: z changes sign

$V = (1, 0, 0; 0, 1, 0; 0, 0, -1)$

b)

U first, so U goes on the right: VU

$VU = (1, 0, 0; 0, 1, 0; 0, 0, -1)(1, 0, 0; 0, 0, -1; 0, 1, 0)$

$\boxed{VU = (1, 0, 0; 0, 0, -1; 0, -1, 0)}$

c)

$VU(2; -1; 3) = (2; 0 \times (-1) + (-1) \times 3; -1 \times (-1) + 0 \times 3)$

$\boxed{(2, -3, 1)}$

d)

$UV = (1, 0, 0; 0, 0, -1; 0, 1, 0)(1, 0, 0; 0, 1, 0; 0, 0, -1)$

$UV = (1, 0, 0; 0, 0, 1; 0, 1, 0)$

$\boxed{UV \neq VU: \text{the order matters}}$

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10

$$\mathbf{R} = \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

a. Describe fully the transformation represented by **R**

b. Find the image of the point (2, 0, 5) under this transformation.

c. Write down $\mathbf{R}^{-1}$ and describe the transformation it represents.

d. Find the smallest positive integer n for which $\mathbf{R}^n = \mathbf{I}$

(6 marks)

a)

The z row and column are (0, 0, 1): z is unchanged

$\cos\theta = \frac{1}{2}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60^\circ$

Rotation of 60° anticlockwise about the z-axis

b)

$x = \frac{1}{2} \times 2 - \frac{\sqrt{3}}{2} \times 0 = 1$

$y = \frac{\sqrt{3}}{2} \times 2 + \frac{1}{2} \times 0 = \sqrt{3}$

$z = 5$

$(1, \sqrt{3}, 5)$

c)

Undo the rotation: rotate by -60°

Rows separated by ;

$R^{-1} = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0; -\frac{\sqrt{3}}{2}, \frac{1}{2}, 0; 0, 0, 1)$: rotation 60° clockwise about the z-axis

d)

R^n is a rotation of $60n^\circ$ about the z-axis

$60n = 360$

$n = 6$

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11 The parallelogram Q has vertices $O(0, 0)$, $A(2, 0)$, $B(3, 2)$ and $C(1, 2)$, as shown on the grid.

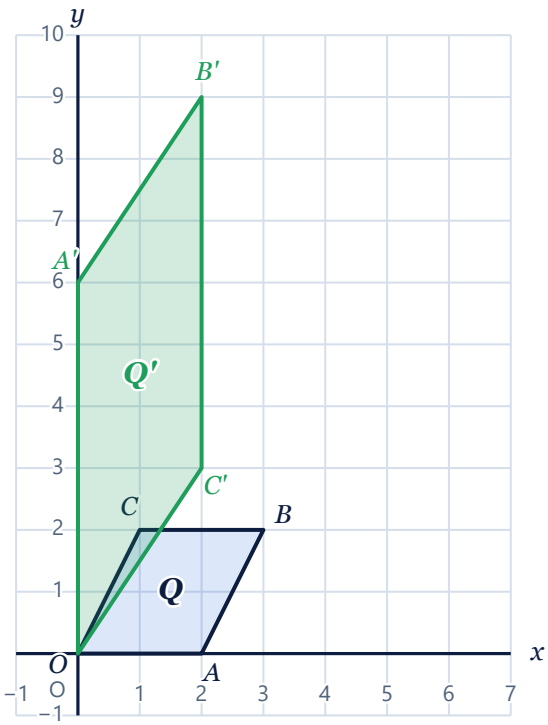
Q is mapped to Q' by a stretch parallel to the x-axis, scale factor 3, followed by a reflection in the line $y = x$

a. Find the single matrix **T** that represents this combined transformation.

b. Find the coordinates of the vertices of Q' and draw Q' on the grid.

c. Find the area of Q'

d. Find $\mathbf{T}^{-1}$ and describe the transformation it represents as a combination of two transformations.



(8 marks)

a)

Rows separated by ;

stretch S = (3, 0; 0, 1)

reflection R = (0, 1; 1, 0)

S first, so it goes on the right

T = RS = (0, 1; 1, 0)(3, 0; 0, 1)

$$\boxed{T = (0, 1; 3, 0)}$$

b)

(0, 1; 3, 0)(0, 2, 3, 1; 0, 0, 2, 2) = (0, 0, 2, 2; 0, 6, 9, 3)

$$\boxed{O(0, 0),\ A'(0, 6),\ B'(2, 9),\ C'(2, 3)}$$

c)

area of Q = 2 × 2 = 4

det T = 0 × 0 − 1 × 3 = −3

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Area scale factor = $|\det T| = 3$

area of Q' = 3×4

area of $Q' = 12$

d)

$$T^{-1} = \frac{1}{-3}\begin{pmatrix} 0 & -1 \\ -3 & 0 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

Undo the steps in reverse order

$$T^{-1} = S^{-1}R^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Reflection in $y = x$, then a stretch parallel to the x-axis, scale factor $\frac{1}{3}$

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12 The transformation T is represented by $\mathbf{M} = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}$

a. Show that $\mathbf{M}^2 = \mathbf{I}$, and explain what this tells you about T.

b. Show that $y = x$ is a line of invariant points of T.

c. Find the invariant lines of T of the form $y = mx + c$, showing that one family of them does not pass through the origin.

(8 marks)

a)

Rows separated by ;

$M^2 = (2, -1; 3, -2)(2, -1; 3, -2)$

$M^2 = (4 - 3, -2 + 2; 6 - 6, -3 + 4)$

$M^2 = (1, 0; 0, 1) = I$

T is self-inverse: doing T twice gets back to the start

b)

$M(x; x) = (2x - x; 3x - 2x)$

$M(x; x) = (x; x)$

Every point on $y = x$ is invariant

c)

General point on the line: $(x, mx + c)$

$x' = 2x - (mx + c) = 2x - mx - c$

$y' = 3x - 2(mx + c) = 3x - 2mx - 2c$

Image must satisfy $y' = mx' + c$

$3x - 2mx - 2c = m(2x - mx - c) + c$

$3x - 2mx - 2c = 2mx - m^2x - mc + c$

Compare the x terms

$3 - 2m = 2m - m^2$

$m^2 - 4m + 3 = 0$

$(m - 1)(m - 3) = 0$

Compare the constants

$-2c = -mc + c$

$c(m - 3) = 0$

$m = 1$: $c = 0$, giving $y = x$

$m = 3$: true for every c

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$y = x$, and every line $y = 3x + c$ (these miss O when $c \neq 0$)

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