Loci and regions in the Argand diagram

Worked solutions

Further Maths topic practice

This topic online

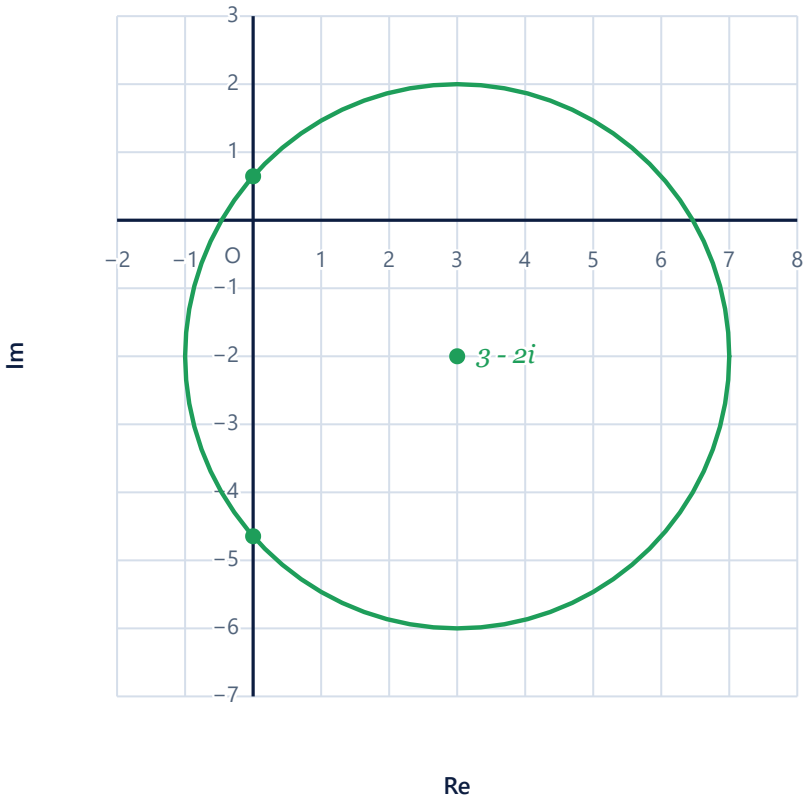
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1 The locus of points z satisfies $|z - 3 + 2i| = 4$

a. Sketch the locus on the Argand diagram.

b. Find a cartesian equation of the locus.

c. Find the exact complex numbers represented by the points where the locus meets the imaginary axis.



(4 marks)

a)

$|z - a| = r$ is a circle, centre a, radius r

$z - 3 + 2i = z - (3 - 2i)$

Centre $3 - 2i$, the point $(3, -2)$

Radius 4

Circle, centre $(3, -2)$, radius 4

b)

Let $z = x + iy$

$|(x - 3) + (y + 2)i| = 4$

$(x - 3)^2 + (y + 2)^2 = 16$

$(x - 3)^2 + (y + 2)^2 = 16$

c)

On the imaginary axis $x = 0$

$(0 - 3)^2 + (y + 2)^2 = 16$

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$$9 + (y+2)^2 = 16$$
$$(y+2)^2 = 7$$
$$y+2 = \pm\sqrt{7}$$
$$y = -2 \pm \sqrt{7}$$

$\boxed{(-2+\sqrt{7})i \text{ and } (-2-\sqrt{7})i}$

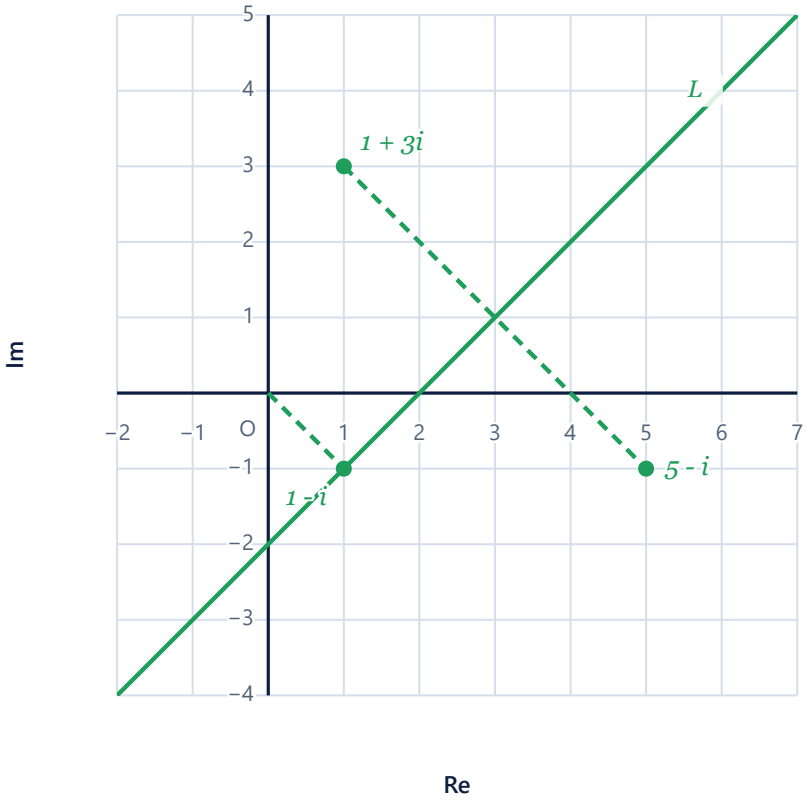
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2 The locus L is given by $|z - 1 - 3i| = |z - 5 + i|$

a. Sketch L on the Argand diagram.

b. Show that a cartesian equation of L is $y = x - 2$

c. Find the complex number on L with the least modulus, and state this least value of $|z|$



(6 marks)

a)

$|z - a| = |z - b|$: perpendicular bisector of a and b

$a = 1 + 3i$, the point $(1, 3)$

$b = 5 - i$, the point $(5, -1)$

Perpendicular bisector of $(1, 3)$ and $(5, -1)$

b)

Let $z = x + iy$

$$|(x - 1) + (y - 3)i| = |(x - 5) + (y + 1)i|$$

$$(x - 1)^2 + (y - 3)^2 = (x - 5)^2 + (y + 1)^2$$

$$x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 - 10x + 25 + y^2 + 2y + 1$$

$$-2x - 6y + 10 = -10x + 2y + 26$$

$$8x - 8y = 16$$

$$x - y = 2$$

$$\boxed{y = x - 2}$$

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c)

Least $|z|$: foot of the perpendicular from O to L

L has gradient 1, so the perpendicular has gradient -1

$$y = -x$$

$$-x = x - 2$$

$$x = 1$$

$$y = -1$$

$$z = 1 - i$$

$$|z| = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$$

$$\boxed{z = 1 - i, \text{ least } |z| = \sqrt{2}}$$

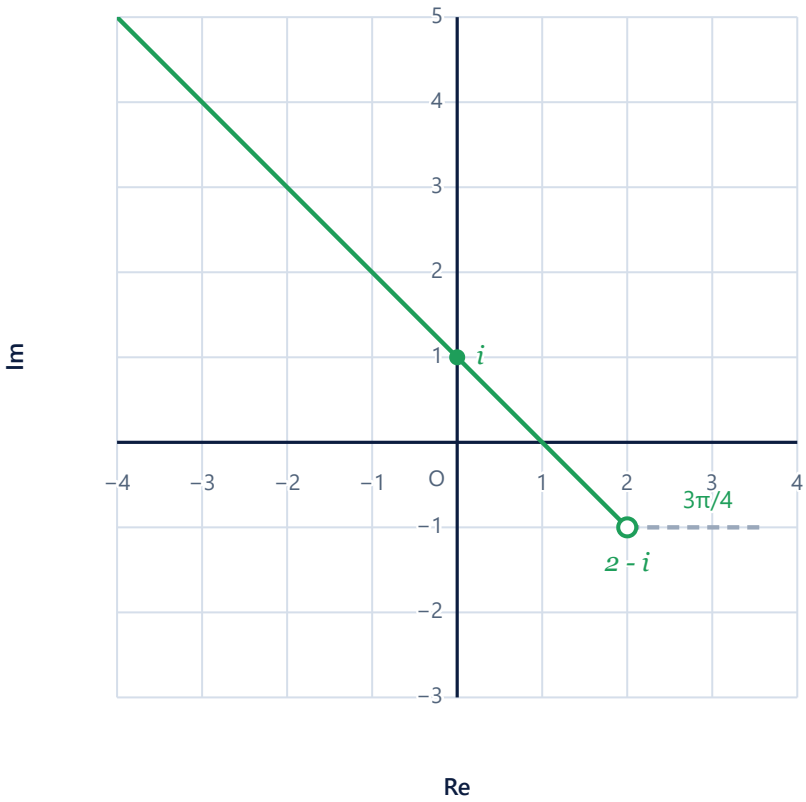
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3 The locus of points z satisfies $\arg(z - 2 + i) = \frac{3\pi}{4}$

a. Sketch the locus on the Argand diagram.

b. Find a cartesian equation of the locus, stating any restriction on x

c. Write down the complex number represented by the point where the locus crosses the imaginary axis.



(4 marks)

a)

arg(z - a) = θ: half-line from a at angle θ to the positive real direction

$z - 2 + i = z - (2 - i)$

Starts at (2, -1), which is not included

$\frac{3\pi}{4}$ is up and to the left at 45°

Half-line from (2, -1) at $\frac{3\pi}{4}$

b)

$\text{gradient} = \tan\frac{3\pi}{4} = -1$

$y - (-1) = -1(x - 2)$

$y + 1 = -x + 2$

$y = 1 - x$

The half-line runs to the left of $x = 2$

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$y = 1 - x, \; x < 2$

c)

On the imaginary axis $x = 0$

$y = 1 - 0 = 1$

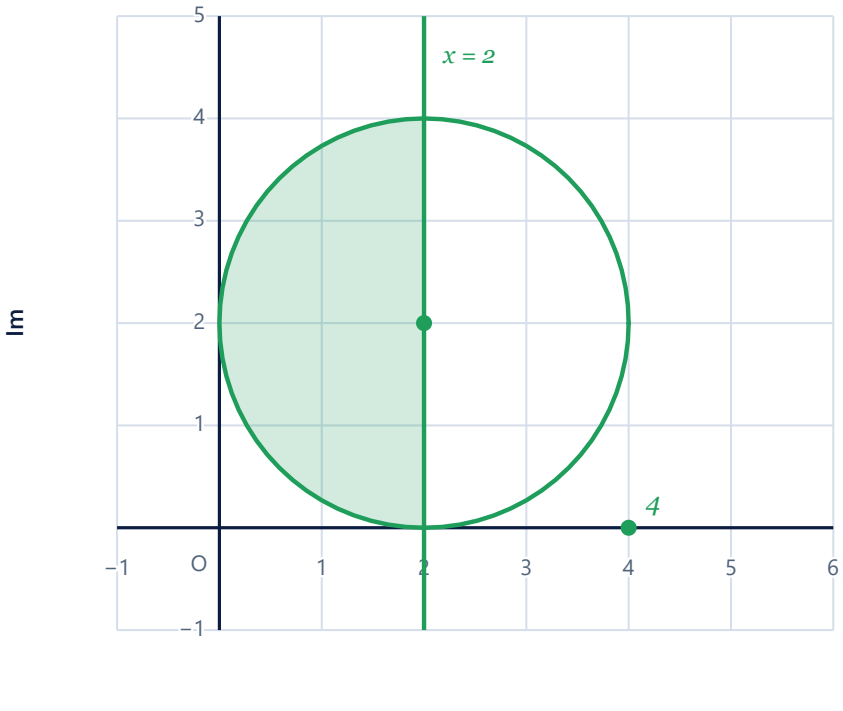
i

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4 a. On the same Argand diagram, sketch the locus $|z - 2 - 2i| = 2$ and the locus $|z| = |z - 4|$

b. Shade on your diagram the region satisfying both $|z - 2 - 2i| \leq 2$ and $|z| \leq |z - 4|$

c. Find the exact area of the shaded region.



(5 marks)

a)

$|z - (2 + 2i)| = 2$: circle, centre (2, 2), radius 2

$|z - 0| = |z - 4|$: perpendicular bisector of 0 and 4

$x = 2$

$x = 2$ passes through the centre of the circle

Circle centre (2, 2), radius 2, and the line $x = 2$

b)

$|z - 2 - 2i| \leq 2$: on or inside the circle

$|z| \leq |z - 4|$: closer to 0 than to 4, so $x \leq 2$

Shade the left half of the disc

c)

Half of a circle of radius 2

$$\text{area} = \frac{1}{2} \times \pi \times 2^2$$

$\text{area} = 2\pi$

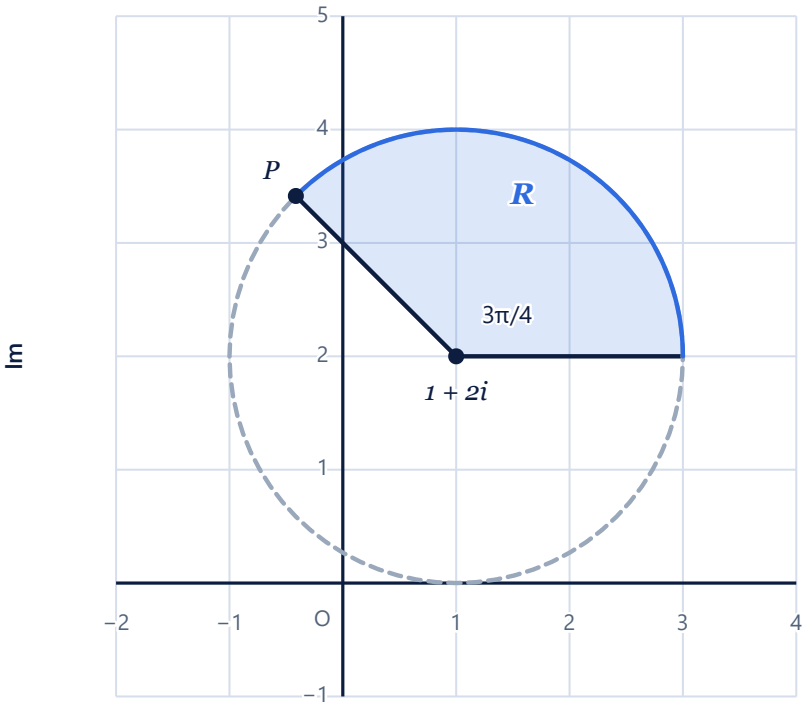
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5 The Argand diagram shows a shaded region R. R is bounded by an arc of the circle with centre $1 + 2i$ and radius 2, and by two straight edges that meet at the centre. One edge is parallel to the real axis. The other edge makes an angle of $\frac{3\pi}{4}$ with the positive real direction.

a. Write down, in terms of z, the two inequalities that define R

b. Find the exact area of R

c. Find, in the form $a + bi$, where a and b are exact, the complex number represented by the vertex P at the end of the sloping edge.



(5 marks)

a)

Distance from the centre $1 + 2i$ is at most 2

$$|z - 1 - 2i| \leq 2$$

Angle measured at the centre, from the positive real direction

$$0 \leq \arg(z - 1 - 2i) \leq \frac{3\pi}{4}$$

$$\boxed{|z - 1 - 2i| \leq 2 \text{ and } 0 \leq \arg(z - 1 - 2i) \leq \frac{3\pi}{4}}$$

b)

R is a sector: area $= \frac{1}{2} r^2 \theta$

$$\text{area} = \frac{1}{2} \times 2^2 \times \frac{3\pi}{4}$$

$$\text{area} = 2 \times \frac{3\pi}{4}$$

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$$\text{area} = \frac{3\pi}{2}$$

c)

P is 2 units from the centre in the direction $\frac{3\pi}{4}$

$$P = 1 + 2i + 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$P = 1 + 2i + 2\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$P = 1 + 2i - \sqrt{2} + \sqrt{2}i$$

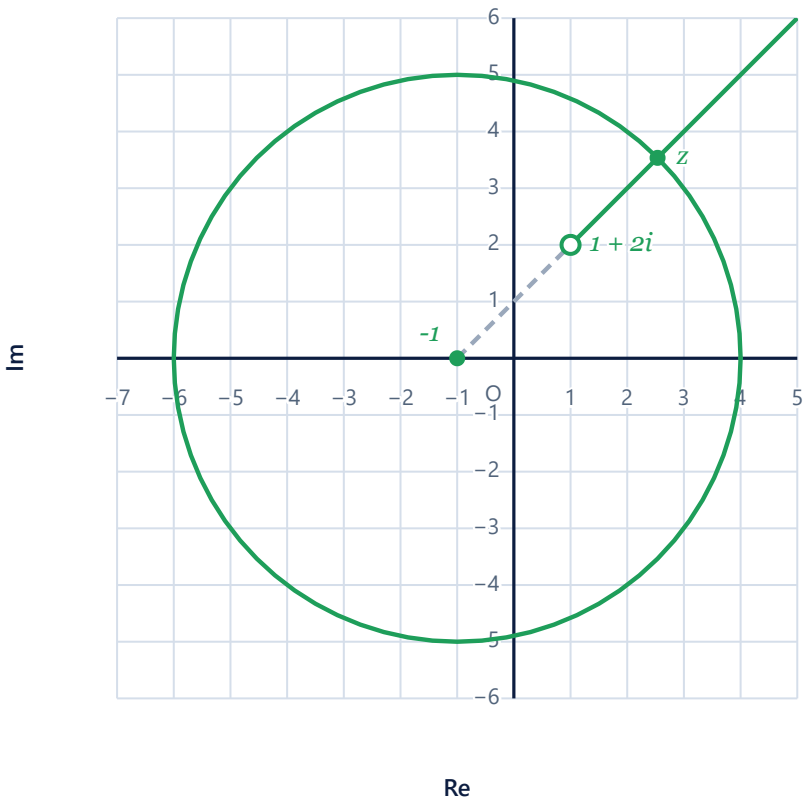
$$P = (1 - \sqrt{2}) + (2 + \sqrt{2})i$$

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6

a. On the same Argand diagram, sketch the locus $|z + 1| = 5$ and the locus $\arg(z - 1 - 2i) = \frac{\pi}{4}$

b. Find, in the form $a + bi$, where a and b are exact, the complex number that satisfies both $|z + 1| = 5$ and $\arg(z - 1 - 2i) = \frac{\pi}{4}$



(7 marks)

a)

$|z - (-1)| = 5$: circle, centre $(-1, 0)$, radius 5

$\arg(z - (1 + 2i)) = \frac{\pi}{4}$: half-line from $(1, 2)$ at $45°$

$(1, 2)$ is $\sqrt{8}$ from $(-1, 0)$, less than 5, so it starts inside the circle

Circle and half-line as shown

b)

Cartesian equation of each locus

$$(x + 1)^2 + y^2 = 25$$

$$y - 2 = \tan\frac{\pi}{4} \times (x - 1)$$

$$y = x + 1, \; x > 1$$

Substitute $y = x + 1$ into the circle

$$(x + 1)^2 + (x + 1)^2 = 25$$

$$2(x + 1)^2 = 25$$

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$$(x+1)^2 = \frac{25}{2}$$

$$x+1 = \pm\frac{5}{\sqrt{2}} = \pm\frac{5\sqrt{2}}{2}$$

$$x = -1 + \frac{5\sqrt{2}}{2} \text{ or } x = -1 - \frac{5\sqrt{2}}{2}$$

The half-line needs $x > 1$

$x = -1 - \frac{5\sqrt{2}}{2} \approx -4.54$: reject

$x = -1 + \frac{5\sqrt{2}}{2} \approx 2.54$

$y = x + 1 = \frac{5\sqrt{2}}{2}$

$$\boxed{z = \left(-1 + \frac{5\sqrt{2}}{2}\right) + \frac{5\sqrt{2}}{2}i}$$

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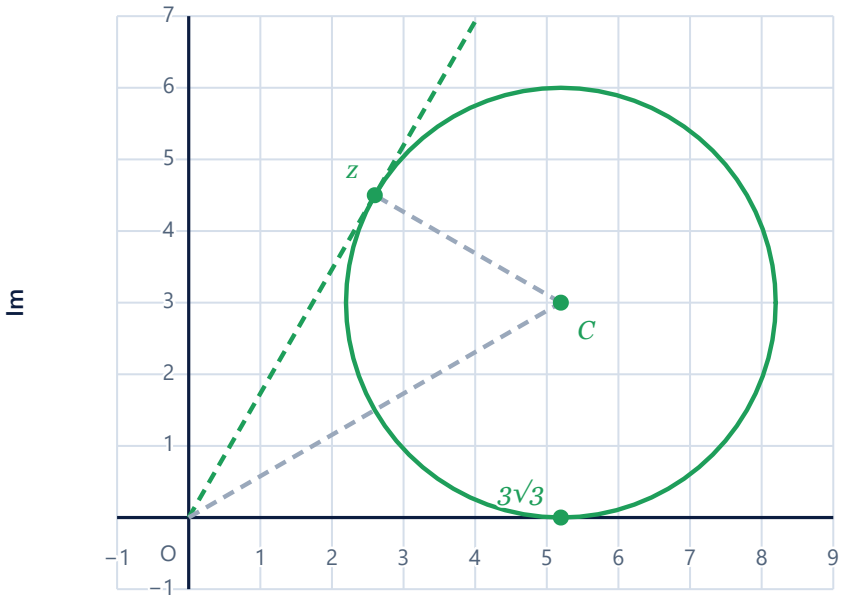
7 The complex number z satisfies $|z - 3\sqrt{3} - 3i| = 3$

a. Sketch the locus of z on the Argand diagram, and show that it touches the real axis.

b. Find the greatest and least values of $|z|$

c. Find the exact greatest value of arg z

d. Find, in the form $a + bi$, where a and b are exact, the value of z for which arg z is greatest.



(9 marks)

a)

Circle, centre C = $3\sqrt{3} + 3i$, radius 3

Centre is 3 units above the real axis

Radius is 3

Distance to the real axis = radius

Touches the real axis at $3\sqrt{3}$

b)

$$|OC| = \sqrt{((3\sqrt{3})^2 + 3^2)}$$

$$|OC| = \sqrt{(27 + 9)} = 6$$

Greatest and least $|z|$ lie on the line through O and C

greatest $|z| = 6 + 3 = 9$

least $|z| = 6 - 3 = 3$

greatest 9, least 3

c)

Greatest arg z: the tangent from O on the upper side

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$$\arg(OC) = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\arg(OC) = \frac{\pi}{6}$$

Radius ⊥ tangent: right-angled triangle, hypotenuse OC = 6

$$\sin\alpha = \frac{3}{6} = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$\text{greatest arg } z = \frac{\pi}{6} + \frac{\pi}{6}$$

$$\boxed{\text{greatest arg } z = \frac{\pi}{3}}$$

d)

Pythagoras for the tangent length

$$|z| = \sqrt{(6^2 - 3^2)} = \sqrt{27} = 3\sqrt{3}$$

$$z = 3\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z = 3\sqrt{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\boxed{z = \frac{3\sqrt{3}}{2} + \frac{9}{2}i}$$

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8 The loci C and L are given by

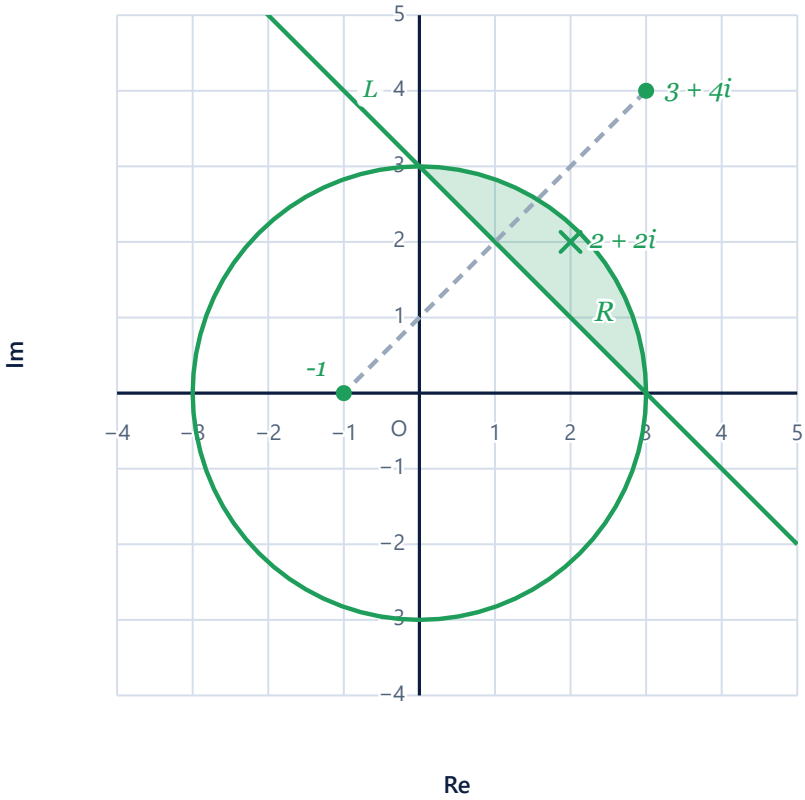
$$C: |z| = 3 \qquad L: |z - 3 - 4i| = |z + 1|$$

a. Show that a cartesian equation of L is $x + y = 3$

b. Find the complex numbers represented by the points where C and L meet.

c. On the Argand diagram, sketch C and L, and shade the region R satisfying both $|z| \leq 3$ and $|z - 3 - 4i| \leq |z + 1|$

d. Determine whether the point representing $2 + 2i$ lies in R



(8 marks)

a)

Let $z = x + iy$

$$(x - 3)^2 + (y - 4)^2 = (x + 1)^2 + y^2$$

$$x^2 - 6x + 9 + y^2 - 8y + 16 = x^2 + 2x + 1 + y^2$$

$$-6x - 8y + 25 = 2x + 1$$

$$24 = 8x + 8y$$

$$\boxed{x + y = 3}$$

b)

$$x^2 + y^2 = 9$$

$$y = 3 - x$$

Substitute

$$x^2 + (3 - x)^2 = 9$$

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$$x^2 + 9 - 6x + x^2 = 9$$
$$2x^2 - 6x = 0$$
$$2x(x - 3) = 0$$
$$x = 0 \text{ or } x = 3$$

$x = 0$: $y = 3$

$x = 3$: $y = 0$

3i and 3

c)

C: circle, centre O, radius 3

L: perpendicular bisector of 3 + 4i and −1

$|z - 3 - 4i| \leq |z + 1|$: the side of L nearer 3 + 4i

That is $x + y \geqslant 3$

R: the part of the disc on or above $x + y = 3$

d)

$|2 + 2i| = \sqrt{(4 + 4)} = \sqrt{8} \approx 2.83$

$\sqrt{8} \leqslant 3$, so inside C

$x + y = 2 + 2 = 4$

$4 \geqslant 3$, so on the 3 + 4i side of L

Yes, 2 + 2i lies in R

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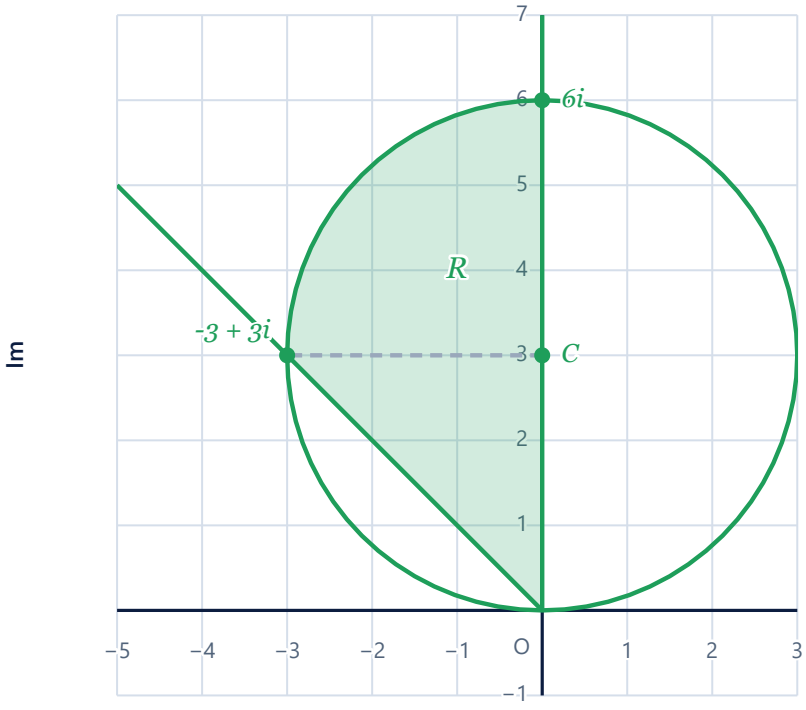
9 The region R in the Argand diagram is defined by

$$R = \{z : |z - 3i| \leq 3\} \cap \{z : \frac{\pi}{2} \leq \arg z \leq \frac{3\pi}{4}\}$$

a. Show, by shading on the Argand diagram, the region R

b. Show that the half-line $\arg z = \frac{3\pi}{4}$ meets the circle $|z - 3i| = 3$ at the point representing $-3 + 3i$

c. Find the exact area of R, giving your answer in the form $p(\pi + q)$, where p and q are rational.



(8 marks)

a)

$|z - 3i| \leq 3$: disc, centre C(0, 3), radius 3, through O

$\arg z = \frac{\pi}{2}$: the positive imaginary axis

$\arg z = \frac{3\pi}{4}$: half-line from O, up and to the left at 45°

Shade the part of the disc between the two half-lines

b)

$$\arg z = \frac{3\pi}{4}: y = -x, x < 0$$

$$x^2 + (y - 3)^2 = 9$$

Substitute $y = -x$

$$x^2 + (-x - 3)^2 = 9$$

$$x^2 + x^2 + 6x + 9 = 9$$

$$2x^2 + 6x = 0$$

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$$2x(x+3) = 0$$

$x = 0$ is O, not on the half-line

$x = -3$, $y = 3$

The point $-3 + 3i$

c)

Join C(0, 3) to P(−3, 3): CP is horizontal, OC is vertical

Triangle OCP: right angle at C, OC = CP = 3

$$\text{triangle} = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}$$

Sector at C from CP up to 6i: angle $\frac{\pi}{2}$

$$\text{sector} = \frac{1}{2} \times 3^2 \times \frac{\pi}{2} = \frac{9\pi}{4}$$

$$\text{area} = \frac{9\pi}{4} + \frac{9}{2}$$

$\text{area} = \frac{9}{4}(\pi + 2)$

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10 The complex number z satisfies $\arg(z - 2) = \frac{2\pi}{3}$

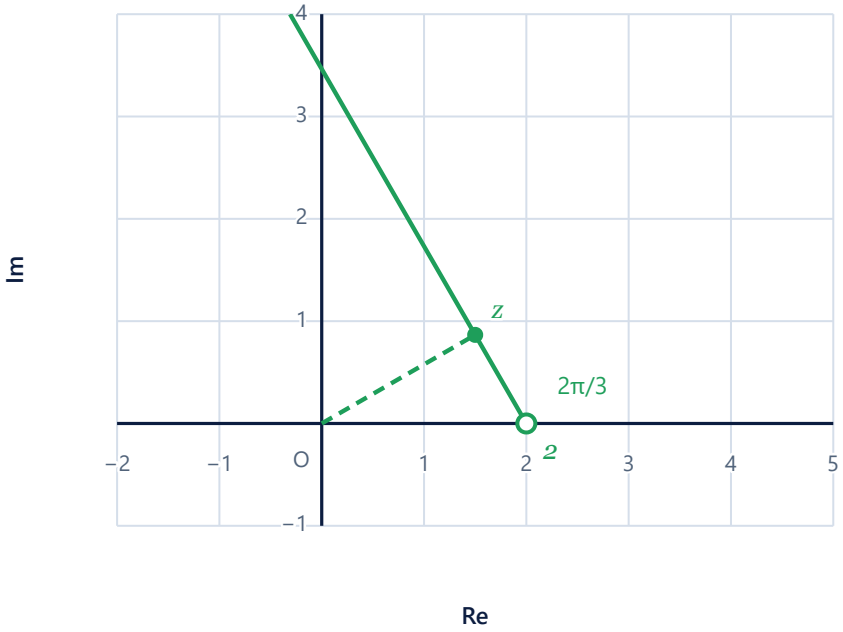
a. Sketch the locus of z on the Argand diagram.

b. Find a cartesian equation of the locus, stating any restriction on x

c. Find the exact least value of $|z|$ and the complex number z at which it occurs.

The complex number w satisfies $\arg(w - 2) = \frac{\pi}{3}$

d. Explain why $|w| > 2$ for every such w, and why $|w|$ has no least value.



(8 marks)

a)

Half-line from 2 at $\frac{2\pi}{3}$: up and to the left, 60° above the negative real direction

The point 2 is not included

Half-line from (2, 0) at $\frac{2\pi}{3}$

b)

$$\text{gradient} = \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$y - 0 = -\sqrt{3}(x - 2)$$

$y = -\sqrt{3}x + 2\sqrt{3}, x < 2$

c)

Least $|z|$: foot of the perpendicular from O, if it lies on the half-line

Perpendicular gradient = $\frac{1}{\sqrt{3}}$

$$y = \frac{x}{\sqrt{3}}$$

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$$\frac{x}{\sqrt{3}} = -\sqrt{3x + 2\sqrt{3}}$$

Multiply by $\sqrt{3}$

$$x = -3x + 6$$

$$4x = 6$$

$$x = \frac{3}{2}$$

$$y = \frac{3}{2} \div \sqrt{3} = \frac{\sqrt{3}}{2}$$

$x = \frac{3}{2} < 2$, so the foot is on the half-line

$$|z| = \sqrt{\left(\frac{9}{4} + \frac{3}{4}\right)} = \sqrt{3}$$

least $|z| = \sqrt{3}$ at $z = \frac{3}{2} + \frac{\sqrt{3}}{2}i$

d)

$\arg(w - 2) = \frac{\pi}{3}$: half-line from 2, up and to the right

So $\text{Re}\, w > 2$ for every w on it

$$|w| = \sqrt{((\text{Re}\, w)^2 + (\text{Im}\, w)^2)} \geqslant \text{Re}\, w > 2$$

As w moves towards 2, $|w|$ gets closer and closer to 2

But $w = 2$ is not on the half-line

$|w| > 2$ always, and no least value is reached

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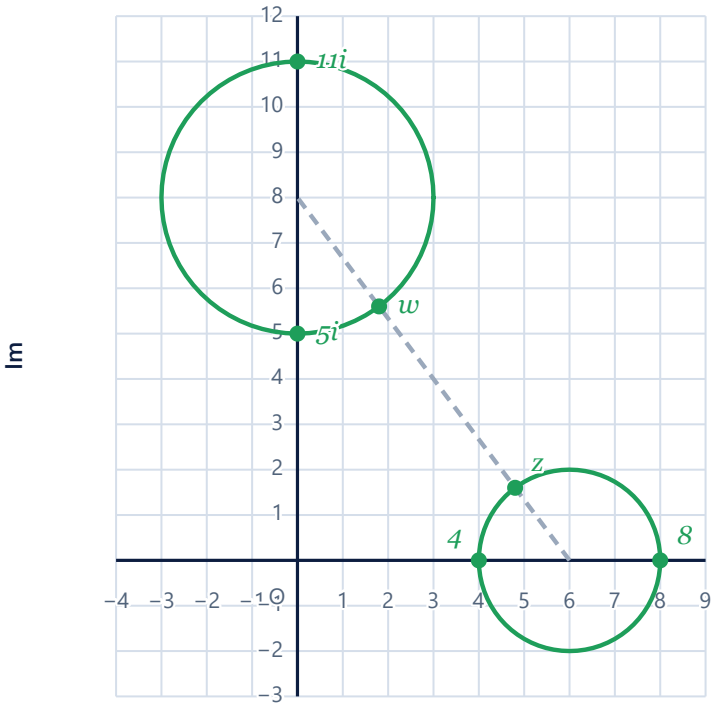
11 The complex numbers z and w satisfy

$$|z - 6| = 2 \quad \text{and} \quad |w - 8i| = 3$$

a. On the same Argand diagram, sketch the locus of z and the locus of w, showing the complex numbers where each locus meets the coordinate axes.

b. Find the range of possible values of $|z - w|$

c. Find the values of z and w for which $|z - w|$ takes its least value.



(9 marks)

a)

$|z - 6| = 2$: circle, centre (6, 0), radius 2

Meets the real axis at $6 - 2 = 4$

and at $6 + 2 = 8$

Radius $2 < 6$, so it does not reach the imaginary axis

$|w - 8i| = 3$: circle, centre (0, 8), radius 3

Meets the imaginary axis at $8 - 3 = 5$

and at $8 + 3 = 11$

Radius $3 < 8$, so it does not reach the real axis

Meets the axes at 4, 8 and 5i, 11i

b)

$|z - w|$ = distance between a point on each circle

distance between centres $= \sqrt{(6^2 + 8^2)} = 10$

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$10 > 2 + 3$, so the circles are separate

least $= 10 - 2 - 3 = 5$

greatest $= 10 + 2 + 3 = 15$

$$\boxed{5 \leqslant |z - w| \leqslant 15}$$

c)

Least distance is along the line of centres

vector from 6 to $8i = -6 + 8i$

unit vector $= \dfrac{(-6 + 8i)}{10} = -0.6 + 0.8i$

$z = 6 + 2(-0.6 + 0.8i)$

$z = 4.8 + 1.6i$

$w = 8i + 3(0.6 - 0.8i)$

$w = 1.8 + 5.6i$

Check: $|z - w| = |3 - 4i| = 5$

$$\boxed{z = 4.8 + 1.6i,\ w = 1.8 + 5.6i}$$

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12 Femi draws a plan of his garden on an Argand diagram. The origin O is a corner of his house, and 1 unit represents 1 metre.

A sprinkler at the point S, representing $6 + 4i$, waters every point within 3 m of S. This is the region W.

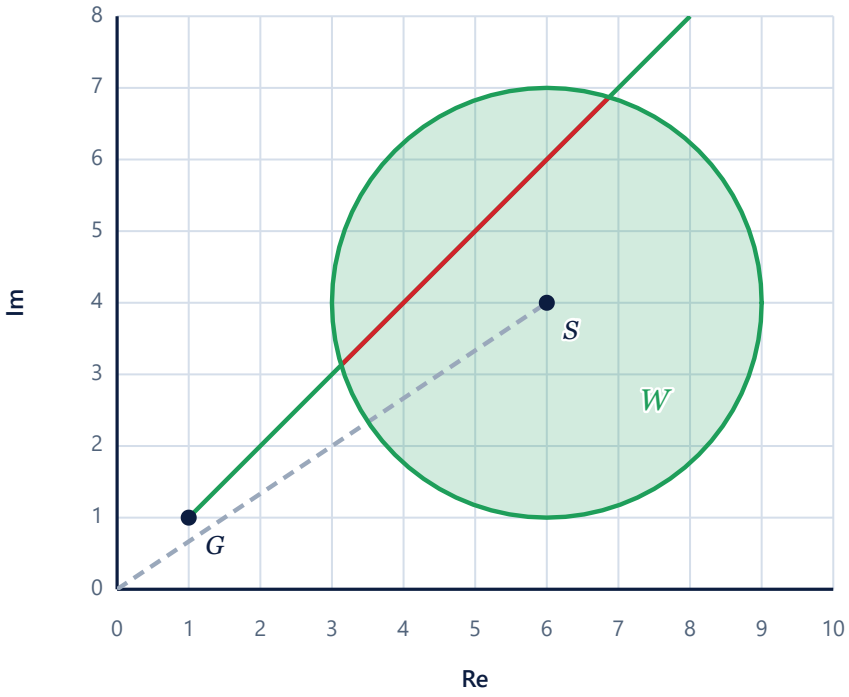
A straight path starts at the gate G, representing $1 + i$, and is modelled by the locus $\arg(z - 1 - i) = \frac{\pi}{4}$

a. Write down an inequality, in terms of z, that defines W

b. Find a cartesian equation of the path.

c. Find the exact length of path that is watered by the sprinkler.

d. Find the shortest distance from the corner of the house to the watered region, giving your answer exactly and to 3 significant figures.



(8 marks)

a)

Within 3 m of 6 + 4i

$$|z - 6 - 4i| \leq 3$$

b)

Half-line from (1, 1) at $\frac{\pi}{4}$

$$\text{gradient} = \tan\frac{\pi}{4} = 1$$

$$y - 1 = 1(x - 1)$$

$$y = x,\ x > 1$$

c)

Edge of W: $(x - 6)^2 + (y - 4)^2 = 9$

Substitute $y = x$

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$$(x-6)^2 + (x-4)^2 = 9$$

$$x^2 - 12x + 36 + x^2 - 8x + 16 = 9$$

$$2x^2 - 20x + 43 = 0$$

$$x = \frac{(20 \pm \sqrt{(400 - 344)})}{4}$$

$$x = \frac{(20 \pm \sqrt{56})}{4}$$

$$x = 5 \pm \frac{\sqrt{14}}{2}$$

$x \approx 3.13$ and $x \approx 6.87$: both > 1, so both on the path

difference in $x = \sqrt{14}$

difference in $y = \sqrt{14}$

length $= \sqrt{(14 + 14)} = \sqrt{28}$

$2\sqrt{7}$ m (≈ 5.29 m)

d)

$$|OS| = |6 + 4i| = \sqrt{(36 + 16)}$$

$$|OS| = \sqrt{52} = 2\sqrt{13} \approx 7.21$$

$7.21 > 3$, so O is outside W

Nearest point of W is on the line OS

distance $= 2\sqrt{13} - 3$

$2\sqrt{13} - 3 \approx 4.21$ m

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