Matrices, determinants and inverses

Worked solutions

Further Maths topic practice

This topic online

1 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} \quad \mathbf{B} = \begin{vmatrix} 1 & 5 \\ -2 & 0 \end{vmatrix}$$

a. Find $2\mathbf{A} - 3\mathbf{B}$

b. Find $\mathbf{AB}$

c. Find $\mathbf{BA}$ and comment on your answers to parts (b) and (c).

(4 marks)

a)

Rows separated by ;

$$2A = (6, -2; 4, 8)$$

$$3B = (3, 15; -6, 0)$$

$$2A - 3B = (6 - 3, -2 - 15; 4 + 6, 8 - 0)$$

$$\boxed{2A - 3B = (3, -17; 10, 8)}$$

b)

Row of A times column of B

$$\text{top left} = 3 \times 1 + (-1) \times (-2) = 5$$

$$\text{top right} = 3 \times 5 + (-1) \times 0 = 15$$

$$\text{bottom left} = 2 \times 1 + 4 \times (-2) = -6$$

$$\text{bottom right} = 2 \times 5 + 4 \times 0 = 10$$

$$\boxed{AB = (5, 15; -6, 10)}$$

c)

$$\text{top left} = 1 \times 3 + 5 \times 2 = 13$$

$$\text{top right} = 1 \times (-1) + 5 \times 4 = 19$$

$$\text{bottom left} = -2 \times 3 + 0 \times 2 = -6$$

$$\text{bottom right} = -2 \times (-1) + 0 \times 4 = 2$$

$$BA = (13, 19; -6, 2)$$

$$\boxed{AB \neq BA: \text{matrix multiplication is not commutative}}$$

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2 The matrices **P** and **Q** are given by

$$\mathbf{P} = \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \end{vmatrix} \quad \mathbf{Q} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 5 \end{vmatrix}$$

a. State the order of the matrix **PQ** and the order of the matrix **QP**

b. Find **PQ**

(3 marks)

a)

Inner numbers must match; outer numbers give the order

PQ: (2 x 3)(3 x 2) gives 2 x 2

QP: (3 x 2)(2 x 3) gives 3 x 3

PQ is 2 x 2, QP is 3 x 3

b)

Row of P times column of Q

top left = $1 \times 2 + 0 \times (-1) + (-2) \times 3 = -4$

top right = $1 \times 1 + 0 \times 0 + (-2) \times 5 = -9$

bottom left = $3 \times 2 + 1 \times (-1) + 4 \times 3 = 17$

bottom right = $3 \times 1 + 1 \times 0 + 4 \times 5 = 23$

Rows separated by ;

PQ = (-4, -9; 17, 23)

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3 $\mathbf{M} = \begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix}$

a. Find det **M**

b. Find $\mathbf{M}^{-1}$

(3 marks)

a)

det = ad − bc

$\det M = 5 \times 2 - 3 \times 4$

$\det M = 10 - 12$

$\det M = -2$

b)

Swap a and d, change the signs of b and c, divide by det

Rows separated by ;

$M^{-1} = \frac{1}{-2}(2, -3; -4, 5)$

$M^{-1} = (-1, \frac{3}{2}; 2, -\frac{5}{2})$

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4 The matrix $\mathbf{A} = \begin{vmatrix} k-3 & 4 \\ 5 & k+5 \end{vmatrix}$, where k is a constant.

a. Find the values of k for which $\mathbf{A}$ is singular.

b. Given that $k = 1$, find $\mathbf{A}^{-1}$

(5 marks)

a)

Singular means det A = 0

$$\det A = (k-3)(k+5) - 4 \times 5$$

$$\det A = k^2 + 2k - 15 - 20$$

$$\det A = k^2 + 2k - 35$$

$$k^2 + 2k - 35 = 0$$

$$(k+7)(k-5) = 0$$

$$\boxed{k = 5 \text{ or } k = -7}$$

b)

Rows separated by ;

$$A = (-2, 4; 5, 6)$$

$$\det A = (-2) \times 6 - 4 \times 5$$

$$\det A = -32$$

Swap a and d, change the signs of b and c, divide by det

$$A^{-1} = \frac{1}{-32}(6, -4; -5, -2)$$

$$\boxed{A^{-1} = \frac{1}{32}(-6, 4; 5, 2)}$$

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5 The matrix $\mathbf{B} = \begin{vmatrix} 2k & 5 \\ k-1 & k+3 \end{vmatrix}$, where k is a real constant.

Show that **B** is non-singular for all real values of k

(4 marks)

Find the determinant

$$\det B = 2k(k+3) - 5(k-1)$$

$$\det B = 2k^2 + 6k - 5k + 5$$

$$\det B = 2k^2 + k + 5$$

Complete the square

$$\det B = 2\left(k^2 + \frac{1}{2}k\right) + 5$$

$$\det B = 2\left(k + \frac{1}{4}\right)^2 - \frac{2}{16} + 5$$

$$\det B = 2\left(k + \frac{1}{4}\right)^2 + \frac{39}{8}$$

$(k + 1/4)^2 \geqslant 0$ for all real k

so $\det B \geqslant \frac{39}{8} > 0$

Or: discriminant $1^2 - 4 \times 2 \times 5 = -39 < 0$, so $\det B$ is never 0

$\det B \neq 0$ for all real k, so B is non-singular

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6 $\mathbf{A} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$ $\mathbf{B} = \begin{vmatrix} 9 & -1 \\ 8 & 3 \end{vmatrix}$

a. Find $\mathbf{A}^{-1}$

b. Find the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$

c. Find the matrix $\mathbf{Y}$ such that $\mathbf{YA} = \mathbf{B}$

(6 marks)

a)

$\det A = 4 \times 2 - 1 \times 3 = 5$

Rows separated by ;

$$\boxed{A^{-1} = \frac{1}{5}(2, -1; -3, 4)}$$

b)

Multiply both sides by A^{-1} on the LEFT

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

$$X = \frac{1}{5}(2, -1; -3, 4)(9, -1; 8, 3)$$

$$X = \frac{1}{5}(18 - 8, -2 - 3; -27 + 32, 3 + 12)$$

$$X = \frac{1}{5}(10, -5; 5, 15)$$

$$\boxed{X = (2, -1; 1, 3)}$$

c)

Multiply both sides by A^{-1} on the RIGHT

$$YAA^{-1} = BA^{-1}$$

$$Y = BA^{-1}$$

$$Y = \frac{1}{5}(9, -1; 8, 3)(2, -1; -3, 4)$$

$$Y = \frac{1}{5}(18 + 3, -9 - 4; 16 - 9, -8 + 12)$$

$$\boxed{Y = \frac{1}{5}(21, -13; 7, 4)}$$

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7 $\mathbf{C} = \begin{vmatrix} k & 3 & 0 \\ 1 & k & 2 \\ -2 & 1 & 1 \end{vmatrix}$, where k is a constant.

a. Show that det $\mathbf{C} = k^2 - 2k - 15$

b. Find the values of k for which $\mathbf{C}$ is singular.

c. Given that $k = 2$, find $\mathbf{C}^{-1}$, showing your working.

(7 marks)

a)

Expand along the top row

$\det C = k(k \times 1 - 2 \times 1) - 3(1 \times 1 - 2 \times (-2)) + 0$

$\det C = k(k - 2) - 3(1 + 4)$

$\det C = k^2 - 2k - 15$

$\det C = k^2 - 2k - 15$ (shown)

b)

$k^2 - 2k - 15 = 0$

$(k - 5)(k + 3) = 0$

$k = 5$ or $k = -3$

c)

$\det C = 4 - 4 - 15 = -15$

Matrix of minors, then the sign pattern (+ − +; − + −; + − +)

Rows separated by ;

minors: (0, 5, 5; 3, 2, 8; 6, 4, 1)

cofactors: (0, −5, 5; −3, 2, −8; 6, −4, 1)

Transpose to get the adjugate

adj C = (0, −3, 6; −5, 2, −4; 5, −8, 1)

Divide by det C

$C^{-1} = \frac{1}{-15}(0, -3, 6; -5, 2, -4; 5, -8, 1)$

$C^{-1} = \frac{1}{15}(0, 3, -6; 5, -2, 4; -5, 8, -1)$

Check: C × C^{-1} gives I

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8 The 2 × 2 matrices **P** and **Q** are such that

$\mathbf{P}^{-1} = \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}$ $\mathbf{Q} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

a. Find $(\mathbf{QP})^{-1}$

b. Find **P**

c. Hence find the column vector **r** such that $\mathbf{QPr} = \begin{vmatrix} 3 \\ 1 \end{vmatrix}$

(6 marks)

a)

Inverse of a product: reverse the order

$(QP)^{-1} = P^{-1}Q^{-1}$

$\det Q = 2 \times 1 - 1 \times 1 = 1$

Rows separated by ;

$Q^{-1} = (1, -1; -1, 2)$

$(QP)^{-1} = (1, 2; 3, 7)(1, -1; -1, 2)$

$(QP)^{-1} = (1 - 2, -1 + 4; 3 - 7, -3 + 14)$

$\boxed{(QP)^{-1} = (-1, 3; -4, 11)}$

b)

P is the inverse of P^{-1}

$\det P^{-1} = 1 \times 7 - 2 \times 3 = 1$

$\boxed{P = (7, -2; -3, 1)}$

c)

Multiply both sides on the left by $(QP)^{-1}$

$r = (QP)^{-1}(3; 1)$

$r = (-1 \times 3 + 3 \times 1; -4 \times 3 + 11 \times 1)$

$\boxed{r = (0; -1)}$

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9 $\mathbf{A} = \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$ and $\mathbf{I}$ is the 2 × 2 identity matrix.

a. Find $\mathbf{A}^2$

b. Find the constants p and q such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

c. Hence, by multiplying your equation by $\mathbf{A}^{-1}$, find $\mathbf{A}^{-1}$

(6 marks)

a)

Rows separated by ;

$A^2 = (3, 2; -1, 1)(3, 2; -1, 1)$

$A^2 = (9 - 2, 6 + 2; -3 - 1, -2 + 1)$

$\boxed{A^2 = (7, 8; -4, -1)}$

b)

$pA + qI = (3p + q, 2p; -p, p + q)$

Compare entries

$2p = 8$

$p = 4$

$3p + q = 7$

$12 + q = 7$

$q = -5$

Check: $-p = -4$ and $p + q = -1$, both match

$\boxed{p = 4, q = -5}$

c)

$A^2 = 4A - 5I$

Multiply through by A^{-1}

$A = 4I - 5A^{-1}$

$5A^{-1} = 4I - A$

$5A^{-1} = (4 - 3, -2; 1, 4 - 1)$

$\boxed{A^{-1} = \frac{1}{5}(1, -2; 1, 3)}$

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10 $\mathbf{M} = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & a \\ 0 & 3 & 1 \end{vmatrix}$, where a is a constant.

a. Find det **M** in terms of a

b. State the value of a for which **M** is singular.

c. Given that $a = 4$, use algebra to find $\mathbf{M}^{-1}$

d. Hence write down $(2\mathbf{M})^{-1}$

(8 marks)

a)

Expand along the top row

$$\det M = 1(1 \times 1 - a \times 3) - 0(2 \times 1 - a \times 0) + 2(2 \times 3 - 1 \times 0)$$

$$\det M = 1 - 3a + 12$$

$$\boxed{\det M = 13 - 3a}$$

b)

$$13 - 3a = 0$$

$$\boxed{a = \frac{13}{3}}$$

c)

$$\det M = 13 - 12 = 1$$

Minors, then the sign pattern (+ - +; - + -; + - +)

Rows separated by ;

minors: (-11, 2, 6; -6, 1, 3; -2, 0, 1)

cofactors: (-11, -2, 6; 6, 1, -3; -2, 0, 1)

Transpose, then divide by det M = 1

$$\boxed{M^{-1} = (-11, 6, -2; -2, 1, 0; 6, -3, 1)}$$

d)

$$(kM)^{-1} = \frac{1}{k}M^{-1}$$

$$(2M)^{-1} = \frac{1}{2}M^{-1}$$

$$\boxed{(2M)^{-1} = \frac{1}{2}(-11, 6, -2; -2, 1, 0; 6, -3, 1)}$$

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11 A conservation group models the numbers of young and adult otters on a stretch of river. The numbers of young otters, y_n, and adult otters, a_n, n years after the survey began satisfy

$$\begin{vmatrix} y_{n+1} \\ a_{n+1} \end{vmatrix} = \mathbf{M} \begin{vmatrix} y_n \\ a_n \end{vmatrix} \quad \text{where} \quad \mathbf{M} = \begin{vmatrix} 0 & 1.2 \\ 0.5 & 0.6 \end{vmatrix}$$

This year there are 300 young otters and 200 adult otters.

a. Use the model to predict the number of young otters and the number of adult otters next year.

b. Find $\mathbf{M}^{-1}$

c. Hence find the numbers of young and adult otters last year, according to the model.

d. The group's records show 247 adult otters last year. Comment on the model in the light of this. (7 marks)

a)

Multiply this year's vector by M

Rows separated by ;

$$\text{next year} = (0, 1.2; 0.5, 0.6)(300; 200)$$

$$\text{young} = 0 \times 300 + 1.2 \times 200 = 240$$

$$\text{adults} = 0.5 \times 300 + 0.6 \times 200 = 270$$

240 young and 270 adults

b)

$$\det M = 0 \times 0.6 - 1.2 \times 0.5 = -0.6$$

$$M^{-1} = \frac{1}{-0.6}(0.6, -1.2; -0.5, 0)$$

$$\boxed{M^{-1} = (-1, 2; \frac{5}{6}, 0)}$$

c)

Go back one year with M^{-1}

$$\text{last year} = (-1, 2; \frac{5}{6}, 0)(300; 200)$$

$$\text{young} = -1 \times 300 + 2 \times 200 = 100$$

$$\text{adults} = \frac{5}{6} \times 300 + 0 \times 200 = 250$$

100 young and 250 adults

d)

Model: 250 adults, records: 247

$$\text{error} = \frac{3}{247} \times 100 = 1.2\%$$

Very close, so the model seems reliable

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12 $\mathbf{Q} = \begin{vmatrix} k & 4 \\ -2 & -k \end{vmatrix}$, where k is a real constant, and $\mathbf{I}$ is the 2 × 2 identity matrix.

a. Show that $\mathbf{Q}^2 = (k^2 - 8)\mathbf{I}$

b. Find the exact values of k for which $\mathbf{Q}$ is singular.

c. Find the values of k for which $\mathbf{Q}$ is self-inverse, that is $\mathbf{Q}^2 = \mathbf{I}$

d. Given that $k = 1$, use part (a) to find $\mathbf{Q}^{-1}$

(7 marks)

a)

Rows separated by ;

$$Q^2 = (k, 4; -2, -k)(k, 4; -2, -k)$$

$$Q^2 = (k^2 - 8, 4k - 4k; -2k + 2k, -8 + k^2)$$

$$Q^2 = (k^2 - 8, 0; 0, k^2 - 8)$$

$$\boxed{Q^2 = (k^2 - 8)I \text{ (shown)}}$$

b)

$$\det Q = k \times (-k) - 4 \times (-2)$$

$$\det Q = 8 - k^2$$

$$8 - k^2 = 0$$

$$k^2 = 8$$

$$\boxed{k = \pm 2\sqrt{2}}$$

c)

$$k^2 - 8 = 1$$

$$k^2 = 9$$

$$\boxed{k = 3 \text{ or } k = -3}$$

d)

$$Q^2 = (1 - 8)I = -7I$$

Multiply both sides by Q^{-1}

$$Q = -7Q^{-1}$$

$$Q^{-1} = -\frac{1}{7}Q$$

$$Q^{-1} = -\frac{1}{7}(1, 4; -2, -1)$$

$$\boxed{Q^{-1} = \frac{1}{7}(-1, -4; 2, 1)}$$

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