Proof by induction

Worked solutions

Further Maths topic practice

This topic online

1 Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (4r - 1) = n(2n + 1)$$

(4 marks)

Basis: $n = 1$

LHS $= 4(1) - 1 = 3$

RHS $= 1(2 + 1) = 3$

LHS = RHS, so true for $n = 1$

Assume true for $n = k$

$\Sigma(r = 1 \text{ to } k)(4r - 1) = k(2k + 1)$

$n = k + 1$: add the $(k + 1)$th term

$$\begin{aligned}\Sigma(r = 1 \text{ to } k + 1)(4r - 1) &= k(2k + 1) + 4(k + 1) - 1 \\ &= 2k^2 + k + 4k + 3 \\ &= 2k^2 + 5k + 3 \\ &= (k + 1)(2k + 3) \\ &= (k + 1)(2(k + 1) + 1)\end{aligned}$$

This is the formula with $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

$\Sigma(4r - 1) = n(2n + 1)$ for all $n \geq 1$

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2 Prove by induction that, for $n \in \mathbb{Z}^+$, $11^n - 6$ is divisible by 5 (4 marks)

Let $f(n) = 11^n - 6$

Basis: $n = 1$

$f(1) = 11 - 6 = 5$

5 is divisible by 5, so true for $n = 1$

Assume true for $n = k$: $f(k) = 11^k - 6$ is divisible by 5

Consider $f(k + 1) - f(k)$

$$f(k+1) - f(k) = (11^{k+1} - 6) - (11^k - 6)$$
$$= 11 \times 11^k - 11^k$$
$$= 10 \times 11^k$$
$$f(k+1) = f(k) + 10 \times 11^k$$

$f(k)$ is divisible by 5 (assumption) and $10 \times 11^k = 5(2 \times 11^k)$ is divisible by 5

So $f(k + 1)$ is divisible by 5: true for $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

$11^n - 6$ is divisible by 5 for all $n \geq 1$

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3 A sequence is defined by

$u_1 = 3, \quad u_{n+1} = 2u_n - 1$

Prove by induction that, for $n \in \mathbb{Z}^+$, $u_n = 2^n + 1$ (4 marks)

Basis: $n = 1$

$2^1 + 1 = 3$

$u_1 = 3$, so true for $n = 1$

Assume true for $n = k$

$u_k = 2^k + 1$

$n = k + 1$: use the recurrence

$$\begin{aligned} u_{k+1} &= 2u_k - 1 \\ &= 2(2^k + 1) - 1 \\ &= 2^{k+1} + 2 - 1 \\ &= 2^{k+1} + 1 \end{aligned}$$

This is the formula with $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

$u_n = 2^n + 1$ for all $n \geq 1$

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4 Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$$

(5 marks)

Basis: $n = 1$

$$\text{LHS} = 1 \times 3 = 3$$

$$\text{RHS} = \frac{1}{6} \times 1 \times 2 \times 9 = 3$$

LHS = RHS, so true for $n = 1$

Assume true for $n = k$

$$\Sigma(r = 1 \text{ to } k)\, r(r+2) = \frac{1}{6}k(k+1)(2k+7)$$

$n = k + 1$: add the $(k+1)$th term

$$\Sigma(r = 1 \text{ to } k+1)\, r(r+2) = \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

Take out the common factor $\frac{1}{6}(k+1)$

$$= \frac{1}{6}(k+1)[k(2k+7) + 6(k+3)]$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18)$$

$$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$$= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+7)$$

This is the formula with $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

$$\boxed{\Sigma r(r+2) = \frac{1}{6}n(n+1)(2n+7)}$$

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5 Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}^n = \begin{vmatrix} 1 & 0 \\ 4^n - 1 & 4^n \end{vmatrix}$$

(5 marks)

Matrices written as rows separated by ;

Let M = (1, 0; 3, 4)

Basis: n = 1

RHS = $(1, 0; 4^1 - 1, 4^1) = (1, 0; 3, 4) = M$

So true for n = 1

Assume true for n = k

$M^k = (1, 0; 4^k - 1, 4^k)$

n = k + 1: $M^{k+1} = M^k M$

$M^{k+1} = (1, 0; 4^k - 1, 4^k)(1, 0; 3, 4)$

Top row: 1 × 1 + 0 × 3 and 1 × 0 + 0 × 4

Bottom row: $(4^k - 1) \times 1 + 4^k \times 3$ and $(4^k - 1) \times 0 + 4^k \times 4$

$M^{k+1} = (1, 0; 4^k - 1 + 3 \times 4^k, 4 \times 4^k)$

$M^{k+1} = (1, 0; 4 \times 4^k - 1, 4^{k+1})$

$M^{k+1} = (1, 0; 4^{k+1} - 1, 4^{k+1})$

This is the formula with n = k + 1

Conclusion

The result is true for n = 1.

If it is true for n = k, then it is true for n = k + 1.

So, by mathematical induction, it is true for all positive integers n.

$\boxed{M^n = (1, 0; 4^n - 1, 4^n)}$

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6 Prove by induction that, for $n \in \mathbb{Z}^+$,

$f(n) = 7^{2n-1} + 5$

is divisible by 12

(6 marks)

Basis: $n = 1$

$f(1) = 7^1 + 5 = 12$

12 is divisible by 12, so true for $n = 1$

Assume true for $n = k$: $f(k) = 7^{2k-1} + 5$ is divisible by 12

$n = k + 1$

$$f(k+1) = 7^{2(k+1)-1} + 5$$
$$= 7^{2k+1} + 5$$
$$= 7^2 \times 7^{2k-1} + 5$$
$$= 49 \times 7^{2k-1} + 5$$

Write it in terms of $f(k)$

$$= 49(7^{2k-1} + 5) - 245 + 5$$
$$= 49f(k) - 240$$
$$= 49f(k) - 12 \times 20$$

$49f(k)$ is divisible by 12 (assumption) and 12×20 is divisible by 12

So $f(k+1)$ is divisible by 12: true for $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

$f(n)$ is divisible by 12 for all $n \geq 1$

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7 Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)} = \frac{n}{3(2n+3)}$$

(6 marks)

Basis: $n = 1$

$$\text{LHS} = \frac{1}{(3 \times 5)} = \frac{1}{15}$$

$$\text{RHS} = \frac{1}{(3 \times 5)} = \frac{1}{15}$$

LHS = RHS, so true for $n = 1$

Assume true for $n = k$

$$\Sigma(r = 1 \text{ to } k) \frac{1}{((2r+1)(2r+3))} = \frac{k}{(3(2k+3))}$$

$n = k + 1$: the $(k+1)$th term has $2r + 1 = 2k + 3$ and $2r + 3 = 2k + 5$

$$\Sigma(r = 1 \text{ to } k+1) \frac{1}{((2r+1)(2r+3))} = \frac{k}{(3(2k+3))} + \frac{1}{((2k+3)(2k+5))}$$

Common denominator $3(2k+3)(2k+5)$

$$= \frac{(k(2k+5)+3)}{(3(2k+3)(2k+5))}$$

$$= \frac{(2k^2+5k+3)}{(3(2k+3)(2k+5))}$$

$$= \frac{((k+1)(2k+3))}{(3(2k+3)(2k+5))}$$

$$= \frac{(k+1)}{(3(2k+5))}$$

$$= \frac{(k+1)}{(3(2(k+1)+3))}$$

This is the formula with $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

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$$\Sigma = \frac{n}{(3(2n+3))}$$

8 Prove by induction that, for all positive **odd** integers n,

$f(n) = 3^n + 5^n$

is divisible by 8

(6 marks)

Basis: $n = 1$

$f(1) = 3 + 5 = 8$

8 is divisible by 8, so true for $n = 1$

Assume true for $n = k$, where k is odd

$f(k) = 3^k + 5^k$ is divisible by 8

The next odd integer is $k + 2$, so consider $f(k + 2)$

$f(k + 2) = 3^{k+2} + 5^{k+2}$

$= 9 \times 3^k + 25 \times 5^k$

Split 25×5^k into $9 \times 5^k + 16 \times 5^k$

$= 9(3^k + 5^k) + 16 \times 5^k$

$= 9f(k) + 8(2 \times 5^k)$

$9f(k)$ is divisible by 8 (assumption) and $8(2 \times 5^k)$ is divisible by 8

So $f(k + 2)$ is divisible by 8: true for $n = k + 2$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 2$.

So, by mathematical induction, it is true for all positive odd integers n.

$3^n + 5^n$ is divisible by 8 for all odd $n \geq 1$

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9 A sequence of positive integers is defined by

$u_1 = 6, \quad u_{n+1} = 3u_n + 4n$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$u_n = 3^{n+1} - 2n - 1$

(6 marks)

Basis: $n = 1$

$3^{1+1} - 2(1) - 1 = 9 - 2 - 1 = 6$

$u_1 = 6$, so true for $n = 1$

Assume true for $n = k$

$u_k = 3^{k+1} - 2k - 1$

$n = k + 1$: use the recurrence with $n = k$

$$\begin{aligned} u_{k+1} &= 3u_k + 4k \\ &= 3(3^{k+1} - 2k - 1) + 4k \\ &= 3^{k+2} - 6k - 3 + 4k \\ &= 3^{k+2} - 2k - 3 \end{aligned}$$

Compare with the formula when $n = k + 1$

$3^{(k+1)+1} - 2(k+1) - 1 = 3^{k+2} - 2k - 3$

These match, so true for $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

$\boxed{u_n = 3^{n+1} - 2n - 1}$

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10 The matrix $\mathbf{M} = \begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix}$

a. Prove by induction that, for $n \in \mathbb{Z}^+$,
$\begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix}^n = \begin{vmatrix} 1+2n & -2n \\ 2n & 1-2n \end{vmatrix}$

b. The transformation represented by $\mathbf{M}^n$ maps the point $A(3, 1)$ to the point $A'(43, 41)$. Find the value of n.

c. A shape S of area 12 cm^2 is transformed by $\mathbf{M}^n$. Show that the area of the image does not depend on n, and state its value.

(9 marks)

a)

Matrices written as rows separated by ;

Basis: $n = 1$

$RHS = (1 + 2, -2; 2, 1 - 2) = (3, -2; 2, -1) = M$

So true for $n = 1$

Assume true for $n = k$

$M^k = (1 + 2k, -2k; 2k, 1 - 2k)$

$n = k + 1$: $M^{k+1} = M^k M$

$M^{k+1} = (1 + 2k, -2k; 2k, 1 - 2k)(3, -2; 2, -1)$

Top row

$3(1 + 2k) + 2(-2k) = 3 + 2k$

$-2(1 + 2k) - 1(-2k) = -2 - 2k$

Bottom row

$3(2k) + 2(1 - 2k) = 2 + 2k$

$-2(2k) - 1(1 - 2k) = -1 - 2k$

$M^{k+1} = (3 + 2k, -2 - 2k; 2 + 2k, -1 - 2k)$

$M^{k+1} = (1 + 2(k + 1), -2(k + 1); 2(k + 1), 1 - 2(k + 1))$

This is the formula with $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

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b)

$$M^n(3, 1) = (3(1 + 2n) - 2n, 3(2n) + (1 - 2n))$$

$$M^n(3, 1) = (3 + 4n, 1 + 4n)$$

$$3 + 4n = 43$$

$$4n = 40$$

$$\boxed{n = 10}$$

Check: $1 + 4 \times 10 = 41$

c)

Area scale factor = $\det M^n$

$$\det M^n = (1 + 2n)(1 - 2n) - (-2n)(2n)$$

$$= 1 - 4n^2 + 4n^2$$

$$= 1$$

The area scale factor is 1 for every n, so the area does not depend on n

$$\boxed{\text{Area} = 12 \text{ cm}^2}$$

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11 Kwame makes a sequence of tile patterns. Pattern 1 uses 5 tiles, and to make pattern $(n + 1)$ from pattern n he adds $4n + 5$ more tiles. So the number of tiles T_n in pattern n satisfies

$T_1 = 5, \quad T_{n+1} = T_n + 4n + 5$

a. Prove by induction that, for $n \in \mathbb{Z}^+$, $T_n = 2n^2 + 3n$

b. One of Kwame's patterns uses exactly 1034 tiles. Find which pattern this is. (7 marks)

a)

Basis: $n = 1$

$2(1)^2 + 3(1) = 5$

$T_1 = 5$, so true for $n = 1$

Assume true for $n = k$

$T_k = 2k^2 + 3k$

$n = k + 1$: use the recurrence

$$T_{k+1} = T_k + 4k + 5$$
$$= 2k^2 + 3k + 4k + 5$$
$$= 2k^2 + 7k + 5$$

Compare with the formula when $n = k + 1$

$2(k + 1)^2 + 3(k + 1) = 2k^2 + 4k + 2 + 3k + 3$

$2(k + 1)^2 + 3(k + 1) = 2k^2 + 7k + 5$

These match, so true for $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

b)

$$2n^2 + 3n = 1034$$
$$2n^2 + 3n - 1034 = 0$$
$$(n - 22)(2n + 47) = 0$$
$$n = 22 \text{ or } n = -\frac{47}{2}$$

n is a positive integer

Pattern 22

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12 The matrix $\mathbf{A} = \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$

a. Find $\mathbf{A}^2$ and $\mathbf{A}^3$

b. Show that the results of part (a) fit the formula

$\mathbf{A}^n = \begin{vmatrix} 2^n & n \times 2^{n-1} \\ 0 & 2^n \end{vmatrix}$

and prove by induction that this formula holds for all $n \in \mathbb{Z}^+$.

c. A triangle of area 3 cm^2 is transformed by $\mathbf{A}^n$ to a triangle of area 3072 cm^2. Find the value of n, and hence write down $\mathbf{A}^n$ for this value.

(10 marks)

a)

Matrices written as rows separated by ;

$A^2 = (2, 1; 0, 2)(2, 1; 0, 2)$

$A^2 = (4 + 0, 2 + 2; 0 + 0, 0 + 4)$

$\boxed{A^2 = (4, 4; 0, 4)}$

$A^3 = (4, 4; 0, 4)(2, 1; 0, 2)$

$A^3 = (8 + 0, 4 + 8; 0, 8)$

$\boxed{A^3 = (8, 12; 0, 8)}$

b)

Check the formula against part (a)

$n = 2$: top right $n \times 2^{n-1} = 2 \times 2 = 4$, diagonal $2^2 = 4$

$n = 3$: top right $n \times 2^{n-1} = 3 \times 4 = 12$, diagonal $2^3 = 8$

Both match part (a)

Basis: $n = 1$

$RHS = (2^1, 1 \times 2^0; 0, 2^1) = (2, 1; 0, 2) = A$

So true for $n = 1$

Assume true for $n = k$

$A^k = (2^k, k \times 2^{k-1}; 0, 2^k)$

$n = k + 1$: $A^{k+1} = A^k A$

$A^{k+1} = (2^k, k \times 2^{k-1}; 0, 2^k)(2, 1; 0, 2)$

Top row

$2^k \times 2 + k \times 2^{k-1} \times 0 = 2^{k+1}$

$2^k \times 1 + k \times 2^{k-1} \times 2 = 2^k + k \times 2^k$

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$$2^k + k \times 2^k = (k+1) \times 2^k$$

Bottom row

$$0 \times 2 + 2^k \times 0 = 0$$

$$0 \times 1 + 2^k \times 2 = 2^{k+1}$$

$$A^{k+1} = (2^{k+1}, (k+1) \times 2^{(k+1)-1}; 0, 2^{k+1})$$

This is the formula with $n = k + 1$

Conclusion

The result is true for $n = 1$.

If it is true for $n = k$, then it is true for $n = k + 1$.

So, by mathematical induction, it is true for all positive integers n.

c)

Area scale factor = det A^n

$$\det A^n = 2^n \times 2^n - 0 = 4^n$$

$$3 \times 4^n = 3072$$

$$4^n = 1024$$

$$n = 5$$

$$A^5 = (2^5, 5 \times 2^4; 0, 2^5)$$

$$\boxed{n = 5, A^5 = (32, 80; 0, 32)}$$

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