Roots of polynomials

Worked solutions

Further Maths topic practice

This topic online

1 The roots of the equation $3x^2 - 7x + 5 = 0$ are α and β.

Without solving the equation, find the exact value of

a. $\alpha^2 + \beta^2$

b. $\frac{1}{\alpha} + \frac{1}{\beta}$

(4 marks)

Sum of roots $= -\frac{b}{a}$, product of roots $= \frac{c}{a}$

$$\alpha + \beta = \frac{7}{3}$$

$$\alpha\beta = \frac{5}{3}$$

a)

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^2 + \beta^2 = \frac{49}{9} - \frac{10}{3}$$

$$\alpha^2 + \beta^2 = \frac{49}{9} - \frac{30}{9}$$

$$\boxed{\alpha^2 + \beta^2 = \frac{19}{9}}$$

b)

Use a common denominator

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{(\alpha + \beta)}{\alpha\beta}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{7}{3} \div \frac{5}{3}$$

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} = \frac{7}{5}}$$

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2 The cubic equation $2x^3 - 5x^2 + 8x - 3 = 0$ has roots α, β and γ.

a. Write down the value of α + β + γ, the value of αβ + βγ + γα and the value of αβγ.

b. Find the value of $\alpha^2 + \beta^2 + \gamma^2$

c. Explain what your answer to part (b) tells you about the roots of the equation.

(5 marks)

a)

$\Sigma\alpha = -\frac{b}{a}, \Sigma\alpha\beta = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}$

$$\alpha + \beta + \gamma = \frac{5}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{8}{2} = 4$$

$$\alpha\beta\gamma = \frac{3}{2}$$

b)

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$\Sigma\alpha^2 = \left(\frac{5}{2}\right)^2 - 2 \times 4$$

$$\Sigma\alpha^2 = \frac{25}{4} - 8$$

$$\boxed{\alpha^2 + \beta^2 + \gamma^2 = -\frac{7}{4}}$$

c)

If α, β, γ were all real then $\alpha^2 + \beta^2 + \gamma^2 \geqslant 0$

But $\alpha^2 + \beta^2 + \gamma^2 < 0$

The roots are not all real (one real root and a complex conjugate pair)

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3 The roots of the equation $x^2 - 4x + 9 = 0$ are α and β.

Without solving the equation, find a quadratic equation with integer coefficients whose roots are $(2\alpha - 1)$ and $(2\beta - 1)$.

(4 marks)

Let $w = 2x - 1$ and make x the subject

$$w = 2x - 1$$

$$x = \frac{(w + 1)}{2}$$

Substitute into $x^2 - 4x + 9 = 0$

$$\left(\frac{(w + 1)}{2}\right)^2 - 4\left(\frac{(w + 1)}{2}\right) + 9 = 0$$

$$\frac{(w + 1)^2}{4} - 2(w + 1) + 9 = 0$$

Multiply through by 4

$$(w + 1)^2 - 8(w + 1) + 36 = 0$$

$$w^2 + 2w + 1 - 8w - 8 + 36 = 0$$

$$\boxed{w^2 - 6w + 29 = 0}$$

Check: new sum $= 2(\alpha + \beta) - 2 = 8 - 2 = 6$

Check: new product $= 4\alpha\beta - 2(\alpha + \beta) + 1 = 36 - 8 + 1 = 29$

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4 $f(z) = z^3 + pz^2 + qz - 51$

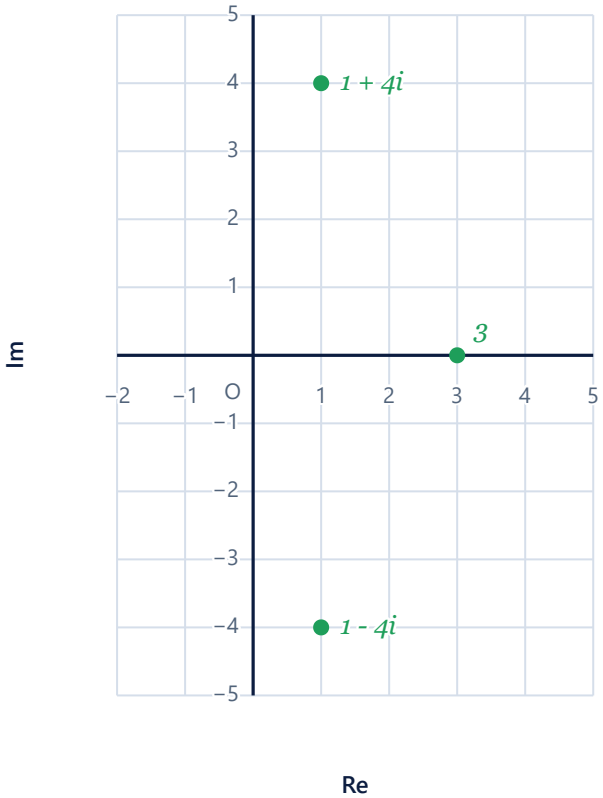
where p and q are real constants.

Given that $1 + 4i$ is a root of the equation $f(z) = 0$,

a. write down another complex root of the equation,

b. find the third root of the equation and the values of p and q,

c. show all three roots of $f(z) = 0$ on a single Argand diagram.



(6 marks)

a) Real coefficients, so complex roots come in conjugate pairs

$\boxed{1 - 4i}$

b) Form the quadratic factor from the conjugate pair

$$\text{sum} = (1 + 4i) + (1 - 4i) = 2$$

$$\text{product} = (1 + 4i)(1 - 4i) = 1 + 16 = 17$$

Quadratic factor: $z^2 - 2z + 17$

$$f(z) = (z^2 - 2z + 17)(z - \gamma)$$

Compare constant terms

$$-17\gamma = -51$$

$$\boxed{\gamma = 3}$$

$$f(z) = (z^2 - 2z + 17)(z - 3)$$

$$f(z) = z^3 - 3z^2 - 2z^2 + 6z + 17z - 51$$

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$f(z) = z^3 - 5z^2 + 23z - 51$

$\boxed{p = -5, q = 23}$

c) Plot (1, 4), (1, −4) and (3, 0)

The two complex roots are reflections of each other in the real axis

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5 The cubic equation $3x^3 - 6x^2 + 2x + 5 = 0$ has roots α, β and γ.

Without solving the equation, find the exact value of

a. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

b. $(\alpha + 3)(\beta + 3)(\gamma + 3)$

c. $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

(7 marks)

$$\Sigma\alpha = \frac{6}{3} = 2$$

$$\Sigma\alpha\beta = \frac{2}{3}$$

$$\alpha\beta\gamma = -\frac{5}{3}$$

a) Common denominator $\alpha\beta\gamma$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{(\beta\gamma + \gamma\alpha + \alpha\beta)}{\alpha\beta\gamma}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{2}{3} \div \left(-\frac{5}{3}\right)$$

$$\boxed{-\frac{2}{5}}$$

b) Expand the brackets

$$(\alpha + 3)(\beta + 3)(\gamma + 3) = \alpha\beta\gamma + 3\Sigma\alpha\beta + 9\Sigma\alpha + 27$$

$$(\alpha + 3)(\beta + 3)(\gamma + 3) = -\frac{5}{3} + 3\left(\frac{2}{3}\right) + 9(2) + 27$$

$$(\alpha + 3)(\beta + 3)(\gamma + 3) = -\frac{5}{3} + 2 + 18 + 27$$

$$\boxed{\frac{136}{3}}$$

c) Square the sum of the reciprocals

$$\Sigma\frac{1}{\alpha^2} = \left(\Sigma\frac{1}{\alpha}\right)^2 - 2\Sigma\frac{1}{\alpha\beta}$$

$$\Sigma\frac{1}{\alpha\beta} = \frac{(\alpha + \beta + \gamma)}{\alpha\beta\gamma}$$

$$\Sigma\frac{1}{\alpha\beta} = 2 \div \left(-\frac{5}{3}\right) = -\frac{6}{5}$$

$$\Sigma\frac{1}{\alpha^2} = \left(-\frac{2}{5}\right)^2 - 2\left(-\frac{6}{5}\right)$$

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$$\Sigma \frac{1}{\alpha^2} = \frac{4}{25} + \frac{60}{25}$$

$$\boxed{\frac{64}{25}}$$

6 The cubic equation $x^3 + 2x^2 - 5x - 3 = 0$ has roots α, β and γ.

Without solving the equation, find the value of

a. $\alpha^2 + \beta^2 + \gamma^2$

b. $\alpha^3 + \beta^3 + \gamma^3$

c. $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta}$

(6 marks)

$$\Sigma\alpha = -2$$

$$\Sigma\alpha\beta = -5$$

$$\alpha\beta\gamma = 3$$

a)

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$\Sigma\alpha^2 = (-2)^2 - 2(-5)$$

$$\boxed{\Sigma\alpha^2 = 14}$$

b) Each root satisfies the equation, so rearrange for x^3

$$\alpha^3 = -2\alpha^2 + 5\alpha + 3$$

Add the same result for β and γ

$$\Sigma\alpha^3 = -2\Sigma\alpha^2 + 5\Sigma\alpha + 3 \times 3$$

$$\Sigma\alpha^3 = -2(14) + 5(-2) + 9$$

$$\Sigma\alpha^3 = -28 - 10 + 9$$

$$\boxed{\Sigma\alpha^3 = -29}$$

c) Common denominator $\alpha\beta\gamma$

$$\frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta} = \frac{(\alpha^2 + \beta^2 + \gamma^2)}{\alpha\beta\gamma}$$

$$\boxed{\frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta} = \frac{14}{3}}$$

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7 The cubic equation $2x^3 + 5x^2 - 4x + 1 = 0$ has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are (2α + 1), (2β + 1) and (2γ + 1), giving your answer in the form $w^3 + aw^2 + bw + c = 0$, where a, b and c are integers to be found. (5 marks)

Let $w = 2x + 1$ and make x the subject

$$x = \frac{(w-1)}{2}$$

Substitute into $2x^3 + 5x^2 - 4x + 1 = 0$

$$2\left(\frac{(w-1)}{2}\right)^3 + 5\left(\frac{(w-1)}{2}\right)^2 - 4\left(\frac{(w-1)}{2}\right) + 1 = 0$$

$$\frac{(w-1)^3}{4} + \frac{5(w-1)^2}{4} - 2(w-1) + 1 = 0$$

Multiply through by 4

$$(w-1)^3 + 5(w-1)^2 - 8(w-1) + 4 = 0$$

$$(w^3 - 3w^2 + 3w - 1) + (5w^2 - 10w + 5) - 8w + 8 + 4 = 0$$

Collect like terms

$$\boxed{w^3 + 2w^2 - 15w + 16 = 0}$$

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8 The cubic equation $2x^3 + px^2 + qx + 6 = 0$, where p and q are constants, has roots α, β and γ.

The cubic equation $w^3 + 4w^2 - 11w + c = 0$, where c is a constant, has roots $(\alpha - 3)$, $(\beta - 3)$ and $(\gamma - 3)$.

Without solving either equation, determine the value of p, the value of q and the value of c. (6 marks)

First equation

$$\Sigma\alpha = -\frac{p}{2}$$

$$\Sigma\alpha\beta = \frac{q}{2}$$

$$\alpha\beta\gamma = -\frac{6}{2} = -3$$

Sum of the new roots = -4

$$(\alpha - 3) + (\beta - 3) + (\gamma - 3) = \Sigma\alpha - 9$$

$$\Sigma\alpha - 9 = -4$$

$$\Sigma\alpha = 5$$

$$-\frac{p}{2} = 5$$

$$\boxed{p = -10}$$

Sum of the new roots in pairs = -11

$$\Sigma(\alpha - 3)(\beta - 3) = \Sigma\alpha\beta - 3(2\Sigma\alpha) + 3 \times 9$$

$$\Sigma\alpha\beta - 6(5) + 27 = -11$$

$$\Sigma\alpha\beta = -8$$

$$\frac{q}{2} = -8$$

$$\boxed{q = -16}$$

Product of the new roots = -c

$$(\alpha - 3)(\beta - 3)(\gamma - 3) = \alpha\beta\gamma - 3\Sigma\alpha\beta + 9\Sigma\alpha - 27$$

$$(\alpha - 3)(\beta - 3)(\gamma - 3) = -3 - 3(-8) + 9(5) - 27$$

$$(\alpha - 3)(\beta - 3)(\gamma - 3) = 39$$

$$-c = 39$$

$$\boxed{c = -39}$$

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9 $f(z) = z^4 + 2z^3 + az^2 + bz + 26$

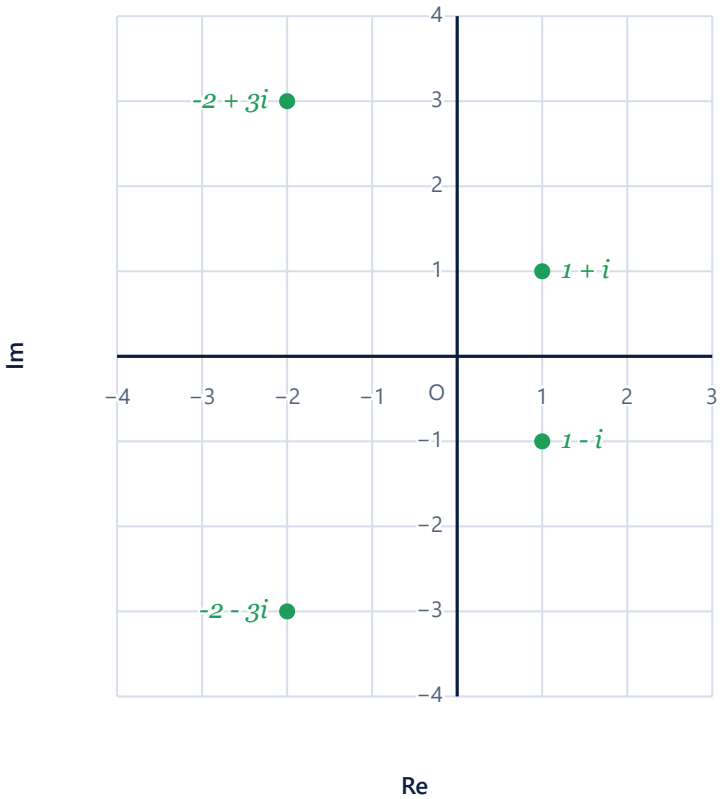
where a and b are real constants.

Given that $-2 + 3i$ is a root of the equation $f(z) = 0$,

a. show that $z^2 + 4z + 13$ is a factor of $f(z)$,

b. find the other three roots of $f(z) = 0$ and the values of a and b,

c. show all four roots of $f(z) = 0$ on a single Argand diagram.



(9 marks)

a) Real coefficients, so the conjugate $-2 - 3i$ is also a root

$$\text{sum} = (-2 + 3i) + (-2 - 3i) = -4$$

$$\text{product} = (-2 + 3i)(-2 - 3i) = 4 + 9 = 13$$

$$(z - (-2 + 3i))(z - (-2 - 3i)) = z^2 + 4z + 13$$

$z^2 + 4z + 13$ is a factor of $f(z)$ (shown)

b) Compare coefficients

$$f(z) = (z^2 + 4z + 13)(z^2 + cz + d)$$

Constant terms

$$13d = 26$$

$$d = 2$$

z^3 terms

$$c + 4 = 2$$

$$c = -2$$

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Solve the other quadratic factor

$$z^2 - 2z + 2 = 0$$

$$(z-1)^2 + 1 = 0$$

$$(z-1)^2 = -1$$

$$z = 1 \pm i$$

Other roots: $-2 - 3i$, $1 + i$, $1 - i$

z^2 terms

$$a = 2 + 4(-2) + 13$$

$$\boxed{a = 7}$$

z terms

$$b = 4(2) + 13(-2)$$

$$\boxed{b = -18}$$

c) Plot $(-2, 3)$, $(-2, -3)$, $(1, 1)$ and $(1, -1)$

Each conjugate pair is symmetric about the real axis

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10 The quartic equation $x^4 - 5x^3 + 2x^2 + 3x - 4 = 0$ has roots α, β, γ and δ.

Without solving the equation, find the value of

a. $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

b. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

c. $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$

d. By considering the value of αβγδ, explain why the equation must have at least two real roots. (9 marks)

$$\Sigma\alpha = 5$$

$$\Sigma\alpha\beta = 2$$

$$\Sigma\alpha\beta\gamma = -3$$

$$\alpha\beta\gamma\delta = -4$$

a)

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$\Sigma\alpha^2 = 25 - 2(2)$$

$$\boxed{\Sigma\alpha^2 = 21}$$

b) Common denominator $\alpha\beta\gamma\delta$

$$\Sigma\frac{1}{\alpha} = \frac{(\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma)}{\alpha\beta\gamma\delta}$$

$$\Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

$$\Sigma\frac{1}{\alpha} = \frac{-3}{-4}$$

$$\boxed{\Sigma\frac{1}{\alpha} = \frac{3}{4}}$$

c) Expand the four brackets

$$(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = \alpha\beta\gamma\delta + \Sigma\alpha\beta\gamma + \Sigma\alpha\beta + \Sigma\alpha + 1$$

$$(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = -4 - 3 + 2 + 5 + 1$$

$$\boxed{(\alpha+1)(\beta+1)(\gamma+1)(\delta+1) = 1}$$

Check: $f(-1) = 1 + 5 + 2 - 3 - 4 = 1$

d) Real coefficients, so non-real roots come in conjugate pairs

$$z \times z^* = |z|^2 > 0$$

If all four roots were non-real, $\alpha\beta\gamma\delta = |z|^2 \times |w|^2 > 0$ for some complex z and w

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But $\alpha\beta\gamma\delta = -4 < 0$

So the roots are not all non-real

Non-real roots come in pairs, so the number of real roots is 0, 2 or 4

It cannot be 0, so it is 2 or 4

So at least two of the roots are real

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11 A storage box is a cuboid. Its length, width and height, in centimetres, are the three roots of the equation

$x^3 - 12x^2 + kx - 28 = 0$

where k is a constant.

a. Explain why the volume of the box is 28 cm^3 and find the total length of all twelve edges of the box.

b. Given that the three dimensions of the box form an arithmetic sequence, find the dimensions of the box and the value of k.

c. Hence write down the total surface area of the box.

d. Chen makes a second box by adding 2 cm to each dimension of the first box. Use a substitution to find the cubic equation, in the form $w^3 + aw^2 + bw + c = 0$, whose roots are the dimensions of the second box, and state the volume of the second box.

(8 marks)

a) Let the roots be α, β, γ

volume $= \alpha\beta\gamma = -\frac{(-28)}{1} = 28\ cm^3$

A cuboid has 4 edges of each dimension

$\alpha + \beta + \gamma = 12$

total edge length $= 4 \times 12$

48 cm

b) Arithmetic sequence: let the roots be $a - d, a, a + d$

$(a - d) + a + (a + d) = 12$

$3a = 12$

$a = 4$

Use the product of the roots

$(4 - d) \times 4 \times (4 + d) = 28$

$16 - d^2 = 7$

$d^2 = 9$

$d = 3$

Dimensions 1 cm, 4 cm, 7 cm

k is the sum of the roots in pairs

$k = 1 \times 4 + 4 \times 7 + 7 \times 1$

$k = 39$

c) Surface area $= 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

surface area $= 2 \times 39$

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$78\ cm^2$

d) Let $w = x + 2$, so $x = w - 2$

$$(w-2)^3 - 12(w-2)^2 + 39(w-2) - 28 = 0$$

$$(w^3 - 6w^2 + 12w - 8) - 12(w^2 - 4w + 4) + 39w - 78 - 28 = 0$$

$$w^3 - 6w^2 + 12w - 8 - 12w^2 + 48w - 48 + 39w - 106 = 0$$

$$w^3 - 18w^2 + 99w - 162 = 0$$

Volume = product of the new roots = 162

$162\ cm^3$

Check: $3 \times 6 \times 9 = 162$

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12 $f(z) = 2z^3 + pz^2 + qz - 30$

where p and q are real constants.

The roots of $f(z) = 0$ are α, β and γ, where $\alpha = 1 - 3i$

a. Find β and γ, where γ is real.

b. Find the value of p and the value of q.

c. Find the cubic equation, in the form $w^3 + aw^2 + bw + c = 0$, whose roots are $(2\alpha - 1)$, $(2\beta - 1)$ and $(2\gamma - 1)$.

d. The roots of the equation in part (c) are represented by the points A, B and C on an Argand diagram. Find the area of triangle ABC.

(10 marks)

a) Real coefficients, so the conjugate is a root

$\boxed{\beta = 1 + 3i}$

$\alpha\beta = (1 - 3i)(1 + 3i) = 1 + 9 = 10$

Product of the roots $= -\frac{d}{a}$

$\alpha\beta\gamma = -\frac{(-30)}{2} = 15$

$10\gamma = 15$

$\boxed{\gamma = \frac{3}{2}}$

b) Build $f(z)$ from its factors

$\alpha + \beta = 2$

Quadratic factor: $z^2 - 2z + 10$

$f(z) = 2(z^2 - 2z + 10)(z - \frac{3}{2})$

$f(z) = (z^2 - 2z + 10)(2z - 3)$

$f(z) = 2z^3 - 3z^2 - 4z^2 + 6z + 20z - 30$

$f(z) = 2z^3 - 7z^2 + 26z - 30$

$\boxed{p = -7, q = 26}$

c) Find the new roots directly

$2\alpha - 1 = 2(1 - 3i) - 1 = 1 - 6i$

$2\beta - 1 = 2(1 + 3i) - 1 = 1 + 6i$

$2\gamma - 1 = 2(\frac{3}{2}) - 1 = 2$

Conjugate pair: sum 2, product $1 + 36 = 37$

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$$(w-2)(w^2-2w+37) = 0$$

$$w^3-2w^2+37w-2w^2+4w-74 = 0$$

$$\boxed{w^3-4w^2+41w-74=0}$$

Check by substituting $x = \frac{(w+1)}{2}$ into $f(x) = 0$ and multiplying by 4

$(w+1)^3-7(w+1)^2+52(w+1)-120$ gives the same cubic

d) A(1, -6), B(1, 6), C(2, 0)

$AB = 6-(-6) = 12$

AB lies on the line $Re = 1$, so the height is the distance from C to that line

$height = 2-1 = 1$

$area = \frac{1}{2} \times 12 \times 1$

$$\boxed{area = 6}$$

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