Simultaneous equations with matrices

Worked solutions

Further Maths topic practice

This topic online

1 Use a matrix method to solve the simultaneous equations

$3x + 2y = 7$

$5x + 4y = 9$

(3 marks)

Write as a matrix equation

Rows separated by ;

$$(3, 2; 5, 4)(x; y) = (7; 9)$$

$$\det = 3 \times 4 - 2 \times 5 = 2$$

$$\text{inverse} = \frac{1}{2}(4, -2; -5, 3)$$

Multiply on the left by the inverse

$$(x; y) = \frac{1}{2}(4, -2; -5, 3)(7; 9)$$

$$(x; y) = \frac{1}{2}(28 - 18; -35 + 27)$$

$$(x; y) = \frac{1}{2}(10; -8)$$

$$\boxed{x = 5, y = -4}$$

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2 Two lines have equations

$4x - 6y = 5$

$6x - 9y = k$

where k is a constant.

a. Show that the matrix of coefficients is singular.

b. Find the value of k for which the equations have infinitely many solutions, and interpret this geometrically.

c. Describe the two lines for any other value of k

(4 marks)

a)

Rows separated by ;

$$\det(4, -6; 6, -9) = 4 \times (-9) - (-6) \times 6$$

$$\det = -36 + 36 = 0$$

det = 0, so singular

b)

Make the x and y parts match

$$\text{first} \div 2: 2x - 3y = 2.5$$

$$\text{second} \div 3: 2x - 3y = \frac{k}{3}$$

Same line when the right-hand sides match

$$\frac{k}{3} = 2.5$$

$$k = 7.5$$

$k = 7.5$: the equations are the same line

c)

$$2x - 3y = 2.5$$

$$2x - 3y = \frac{k}{3}$$

$\frac{k}{3} \neq 2.5$ so the lines are parallel and distinct

Parallel lines: no solutions

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3 Three equations are

$x + 2y - z = -3$

$2x - y + 3z = 14$

$x + y + z = 4$

a. Write the equations in the form **Ax** = **b**

b. Find $\mathbf{A}^{-1}$

c. Hence solve the equations. (5 marks)

a)

Rows separated by ;

$$(1, 2, -1; 2, -1, 3; 1, 1, 1)(x; y; z) = (-3; 14; 4)$$

b)

Calculator (or cofactors)

$$\det A = -5$$

$$A^{-1} = \frac{1}{5}(4, 3, -5; -1, -2, 5; -3, -1, 5)$$

c)

$$(x; y; z) = A^{-1}(-3; 14; 4)$$

$$x = \frac{1}{5}(4 \times (-3) + 3 \times 14 - 5 \times 4) = \frac{1}{5}(10) = 2$$

$$y = \frac{1}{5}(-1 \times (-3) - 2 \times 14 + 5 \times 4) = \frac{1}{5}(-5) = -1$$

$$z = \frac{1}{5}(-3 \times (-3) - 1 \times 14 + 5 \times 4) = \frac{1}{5}(15) = 3$$

Check in the first equation: $2 - 2 - 3 = -3$

$$x = 2, y = -1, z = 3$$

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4 The equations

$$2x + y + kz = 1$$
$$x - y + 2z = 4$$
$$3x + z = 2$$

where k is a constant, do not have a unique solution.

Find the value of k

(3 marks)

No unique solution means det = 0

Rows separated by ;

matrix of coefficients: (2, 1, k; 1, -1, 2; 3, 0, 1)

Expand along the top row

$$\det = 2((-1)(1) - 2 \times 0) - 1(1 \times 1 - 2 \times 3) + k(1 \times 0 - (-1) \times 3)$$

$$\det = -2 + 5 + 3k$$

$$\det = 3 + 3k$$

$$3 + 3k = 0$$

$$\boxed{k = -1}$$

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5 $\mathbf{M} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix}$

a. Show that det **M** = 1 and use algebra to find $\mathbf{M}^{-1}$

b. Hence solve

$x + 2y + z = 3$
$2x + 5y + 2z = 5$
$x + 3y + 2z = 4$

c. Interpret your answer to part (b) geometrically. (6 marks)

a)

Expand along the top row

$\det M = 1(5 \times 2 - 2 \times 3) - 2(2 \times 2 - 2 \times 1) + 1(2 \times 3 - 5 \times 1)$

$\det M = 4 - 4 + 1 = 1$

Minors, then the sign pattern (+ − +; − + −; + − +)

Rows separated by ;

minors: (4, 2, 1; 1, 1, 1; −1, 0, 1)

cofactors: (4, −2, 1; −1, 1, −1; −1, 0, 1)

Transpose and divide by det M = 1

$M^{-1} = (4, -1, -1; -2, 1, 0; 1, -1, 1)$

b)

$(x; y; z) = M^{-1}(3; 5; 4)$

$x = 4 \times 3 - 1 \times 5 - 1 \times 4 = 3$

$y = -2 \times 3 + 1 \times 5 + 0 \times 4 = -1$

$z = 1 \times 3 - 1 \times 5 + 1 \times 4 = 2$

$x = 3, y = -1, z = 2$

c)

Each equation is a plane in 3D

The three planes meet at the single point (3, −1, 2)

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6 Three planes have equations

Π_1: $x + 2y - z = 3$

Π_2: $2x - y + 3z = 1$

Π_3: $4x + 3y + z = 7$

a. Show that the matrix of coefficients is singular.

b. Show that the three equations are consistent, and find the general solution in terms of a parameter t, where $z = t$

c. Describe the geometrical arrangement of the three planes. (6 marks)

a)

Rows separated by ;

matrix: (1, 2, -1; 2, -1, 3; 4, 3, 1)

$\det = 1(-1 \times 1 - 3 \times 3) - 2(2 \times 1 - 3 \times 4) + (-1)(2 \times 3 - (-1) \times 4)$

$\det = -10 + 20 - 10$

$\det = 0$, so singular

b)

Look for a combination of Π_1 and Π_2 that gives Π_3

$2\Pi_1$: $2x + 4y - 2z = 6$

$2\Pi_1 + \Pi_2$: $4x + 3y + z = 7$

This is exactly Π_3, so the system is consistent

Let $z = t$ and solve Π_1 and Π_2

$x + 2y = 3 + t$

$2x - y = 1 - 3t$

Double the second equation and add

$4x - 2y = 2 - 6t$

$5x = 5 - 5t$

$x = 1 - t$

$2y = 3 + t - (1 - t)$

$y = 1 + t$

$x = 1 - t$, $y = 1 + t$, $z = t$

c)

Normals (1, 2, -1), (2, -1, 3), (4, 3, 1): no two parallel

A sheaf: the three planes meet in one line

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7 Three planes have equations

$x - y + 2z = 1$

$3x + y - z = 4$

$5x - y + 3z = k$

where k is a constant.

a. Show that the planes do not meet at a single point.

b. Find the value of k for which the equations are consistent.

c. Describe the arrangement of the planes when k takes this value, and when it takes any other value.

(6 marks)

a)

Rows separated by ;

matrix: (1, -1, 2; 3, 1, -1; 5, -1, 3)

det = 1(1 x 3 - (-1)(-1)) - (-1)(3 x 3 - (-1) x 5) + 2(3 x (-1) - 1 x 5)

det = 2 + 14 - 16 = 0

det = 0, so the planes do not meet at a single point

b)

Find the combination of the first two that matches the left side of the third

$2(x - y + 2z) + (3x + y - z) = 5x - y + 3z$

$2 \times 1 + 4 = 6$

so $5x - y + 3z = 6$ follows from the first two

Consistent when k = 6

c)

Normals (1, -1, 2), (3, 1, -1), (5, -1, 3): no two parallel

k = 6: every point on the line of the first two planes is on the third

k = 6: sheaf (common line); k ≠ 6: triangular prism

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8 Three planes have equations

Π_1: $2x - y + 3z = 4$

Π_2: $4x - 2y + 6z = c$

Π_3: $x + y + z = 2$

where c is a constant.

a. Explain why the equations have no solution when $c = 5$, and describe the arrangement of the planes.

b. Describe the arrangement of the planes when $c = 8$, and state the number of solutions. (4 marks)

a)

Compare normals

$\Pi 2 \div 2$: $2x - y + 3z = \frac{c}{2}$

Same normal $(2, -1, 3)$ as $\Pi 1$, so $\Pi 1$ and $\Pi 2$ are parallel

$c = 5$: $2x - y + 3z = 2.5$

$2.5 \neq 4$, so $\Pi 1$ and $\Pi 2$ are distinct parallel planes

$\Pi 3$ has normal $(1, 1, 1)$, not parallel, so it cuts both

No solution: two parallel planes cut by a third plane

b)

$c = 8$: $2x - y + 3z = 4$

This is $\Pi 1$ again: $\Pi 1$ and $\Pi 2$ coincide

The plane meets $\Pi 3$ in a line: infinitely many solutions

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9 Consider the equations

$x - 2y + 3z = 4$

$2x + y - z = 3$

$kx + 3y - 4z = 5$

where k is a constant.

a. Find the value of k for which the matrix of coefficients is singular.

b. For this value of k, show that the equations are inconsistent.

c. Interpret part (b) geometrically.

(6 marks)

a)

Rows separated by ;

matrix: (1, -2, 3; 2, 1, -1; k, 3, -4)

det = 1(1 x (-4) - (-1) x 3) - (-2)(2 x (-4) - (-1)k) + 3(2 x 3 - 1 x k)

det = -1 + 2(-8 + k) + 3(6 - k)

det = -1 - 16 + 2k + 18 - 3k

det = 1 - k

$k = 1$

b)

Find a combination of the first two that gives the left side of the third

$(2x + y - z) - (x - 2y + 3z) = x + 3y - 4z$

$3 - 4 = -1$

so $x + 3y - 4z = -1$

but the third equation says $x + 3y - 4z = 5$

$-1 \neq 5$, so the equations are inconsistent

c)

Normals (1, -2, 3), (2, 1, -1), (1, 3, -4): no two parallel

The planes form a triangular prism

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10 A café in Leeds sells tea, coffee and slices of cake. Each item has a fixed price. On three days the café recorded these sales.

	Teas	Coffees	Cakes	Takings
Monday	12	20	15	£130.40
Tuesday	18	25	10	£145.10
Wednesday	9	30	22	£174.40

The price of a tea is £t, a coffee £c and a slice of cake £s

a. Write down a matrix equation for t, c and s

b. Use your calculator to find the inverse of the matrix and hence find the price of each item.

c. On Thursday the café sold 10 of each item and took £78.50. Use your answers to comment on the assumption that each item has a fixed price.

(7 marks)

a)

Rows separated by ;

$$(12, 20, 15; 18, 25, 10; 9, 30, 22)(t; c; s) = (130.40; 145.10; 174.40)$$

b)

Calculator: the determinant is non-zero, so the inverse exists

$$\det = 1605$$

$$(t; c; s) = (12, 20, 15; 18, 25, 10; 9, 30, 22)^{-1}(130.40; 145.10; 174.40)$$

$$(t; c; s) = (2.2; 3.1; 2.8)$$

Check Monday: $12 \times 2.20 + 20 \times 3.10 + 15 \times 2.80 = 26.40 + 62 + 42 = 130.40$

tea £2.20, coffee £3.10, cake £2.80

c)

$10(2.20 + 3.10 + 2.80) =$ £81.00

£81.00 predicted, but only £78.50 taken

The prices were not all fixed that day, for example a discount

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11 Ola invests a total of £20 000 in three savings accounts, £x in account X, £y in account Y and £z in account Z. The accounts pay simple interest of 2%, 3.5% and 5% per year respectively.

After one year Ola has earned £640 in interest in total. She invested twice as much in account X as in account Z.

a. Write down three equations in x, y and z

b. Use a matrix method to find how much Ola invested in each account.

c. Her friend Kwame suggests replacing the third condition with "the total interest is 3.2% of the total invested". Explain why the three equations would then not give a unique answer.

(8 marks)

a)

$$x + y + z = 20000$$

$$0.02x + 0.035y + 0.05z = 640$$

x is twice z

$$x - 2z = 0$$

b)

Rows separated by ;

$$(1, 1, 1; 0.02, 0.035, 0.05; 1, 0, -2)(x; y; z) = (20000; 640; 0)$$

Calculator: det = -0.015, non-zero, so the inverse exists

$$(x; y; z) = (1, 1, 1; 0.02, 0.035, 0.05; 1, 0, -2)^{-1}(20000; 640; 0)$$

$$(x; y; z) = (8000; 8000; 4000)$$

Check the interest: 160 + 280 + 200 = 640

£8000 in X, £8000 in Y, £4000 in Z

c)

$$0.032 \times 20000 = 640$$

so the new condition is $0.02x + 0.035y + 0.05z = 640$ again

only two different equations for three unknowns

The matrix is singular: infinitely many solutions, not a unique one

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12 Three planes have equations

$x + y + z = 2$

$2x + ky + 3z = 5$

$-x + 3y + kz = 2$

where k is a constant.

a. Show that the matrix of coefficients is singular when $k^2 - k - 6 = 0$

b. For $k = 3$, show that the planes meet in a line and find the general solution.

c. For $k = -2$, show that the equations are inconsistent and describe the arrangement of the planes.

d. State the arrangement of the planes for any other value of k

(10 marks)

a)

Rows separated by ;

matrix: (1, 1, 1; 2, k, 3; -1, 3, k)

$\det = 1(k \times k - 3 \times 3) - 1(2k - 3 \times (-1)) + 1(2 \times 3 - k \times (-1))$

$\det = k^2 - 9 - 2k - 3 + 6 + k$

$\det = k^2 - k - 6$

singular when $k^2 - k - 6 = 0$ (shown)

$(k - 3)(k + 2) = 0$

$k = 3$ or $k = -2$

b)

$x + y + z = 2$

$2x + 3y + 3z = 5$

Second minus twice the first

$y + z = 1$

$x = 2 - 1 = 1$

Let $z = t$

$y = 1 - t$

Check in the third plane

$-1 + 3(1 - t) + 3t = 2$

true for every t, so all three planes share this line

$x = 1, y = 1 - t, z = t$: a sheaf

c)

$k = -2$: subtract the second equation from the first

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$$(x + y + z) - (2x - 2y + 3z) = -x + 3y - 2z$$
$$2 - 5 = -3$$

so $-x + 3y - 2z = -3$

but the third equation is $-x + 3y - 2z = 2$

Normals $(1, 1, 1)$, $(2, -2, 3)$, $(-1, 3, -2)$: no two parallel

Inconsistent: a triangular prism

d)

$\det \neq 0$: the planes meet at a single point

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