Sums of series

Worked solutions

Further Maths topic practice

This topic online

1 Use the standard results to find the value of

a. $\sum_{r=1}^{40} r^2$

b. $\sum_{r=1}^{15} r^3$

(3 marks)

Standard results: $\sum r = \frac{1}{2}n(n+1)$, $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum r^3 = \frac{1}{4}n^2(n+1)^2$

a)

$$\sum(r = 1 \text{ to } 40)\, r^2 = \frac{1}{6} \times 40 \times 41 \times 81$$

22140

b)

$$\sum(r = 1 \text{ to } 15)\, r^3 = \frac{1}{4} \times 15^2 \times 16^2$$

$$\sum(r = 1 \text{ to } 15)\, r^3 = \frac{1}{4} \times 225 \times 256$$

14400

2 Find the value of $\sum_{r=1}^{12} (2r^2 - 5)$

(3 marks)

Split the sum

$$\sum(2r^2 - 5) = 2\sum r^2 - \sum 5$$

$$\sum(r = 1 \text{ to } 12)\, r^2 = \frac{1}{6} \times 12 \times 13 \times 25 = 650$$

The constant 5 is added 12 times

$$\sum(r = 1 \text{ to } 12)\, 5 = 12 \times 5 = 60$$

$$\sum(2r^2 - 5) = 2 \times 650 - 60$$

1240

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3 Find the value of $\sum_{r=11}^{30} r^3$ (3 marks)

Subtract the sum up to 10, not up to 11

$$\Sigma(r = 11 \text{ to } 30)\, r^3 = \Sigma(r = 1 \text{ to } 30)\, r^3 - \Sigma(r = 1 \text{ to } 10)\, r^3$$

$$\Sigma(r = 1 \text{ to } 30)\, r^3 = \frac{1}{4} \times 30^2 \times 31^2 = 216225$$

$$\Sigma(r = 1 \text{ to } 10)\, r^3 = \frac{1}{4} \times 10^2 \times 11^2 = 3025$$

$$\Sigma(r = 11 \text{ to } 30)\, r^3 = 216225 - 3025$$

$$\boxed{213200}$$

4 Show that $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$ (4 marks)

Expand, then use the standard results

$$\Sigma r(r+3) = \Sigma r^2 + 3\Sigma r$$

$$\Sigma r(r+3) = \frac{1}{6}n(n+1)(2n+1) + 3 \times \frac{1}{2}n(n+1)$$

Take out the common factor $\frac{1}{6}n(n+1)$

$$\Sigma r(r+3) = \frac{1}{6}n(n+1)[(2n+1) + 9]$$

$$\Sigma r(r+3) = \frac{1}{6}n(n+1)(2n+10)$$

$$\Sigma r(r+3) = \frac{1}{6}n(n+1) \times 2(n+5)$$

$$\boxed{\Sigma r(r+3) = \frac{1}{3}n(n+1)(n+5)}$$

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5

a. Show that $\sum_{r=1}^{n} (3r-1)^2 = \frac{1}{2}n(6n^2+3n-1)$

b. Hence find the value of $\sum_{r=1}^{20} (3r-1)^2$

(5 marks)

a)

Expand the bracket first

$$(3r-1)^2 = 9r^2 - 6r + 1$$

$$\Sigma(3r-1)^2 = 9\Sigma r^2 - 6\Sigma r + \Sigma 1$$

$$\Sigma(3r-1)^2 = 9 \times \frac{1}{6}n(n+1)(2n+1) - 6 \times \frac{1}{2}n(n+1) + n$$

$$\Sigma(3r-1)^2 = \frac{3}{2}n(n+1)(2n+1) - 3n(n+1) + n$$

Take out $\frac{1}{2}n$

$$\Sigma(3r-1)^2 = \frac{1}{2}n[3(n+1)(2n+1) - 6(n+1) + 2]$$

$$\Sigma(3r-1)^2 = \frac{1}{2}n[6n^2 + 9n + 3 - 6n - 6 + 2]$$

$$\boxed{\Sigma(3r-1)^2 = \frac{1}{2}n(6n^2+3n-1)}$$

b)

Put $n = 20$ into the formula

$$\Sigma(r=1 \text{ to } 20) = \frac{1}{2} \times 20 \times (6 \times 400 + 60 - 1)$$

$$\Sigma(r=1 \text{ to } 20) = 10 \times 2459$$

$$\boxed{24590}$$

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6

a. Show that $\sum_{r=1}^{n}(r-1)r(r+1) = \frac{1}{4}(n-1)n(n+1)(n+2)$

b. Hence find the value of $\sum_{r=11}^{30}(r-1)r(r+1)$

(6 marks)

a)

$$(r-1)r(r+1) = r(r^2-1) = r^3 - r$$

$$\Sigma(r^3 - r) = \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$$

Take out $\frac{1}{4}n(n+1)$

$$\Sigma(r^3 - r) = \frac{1}{4}n(n+1)[n(n+1) - 2]$$

$$\Sigma(r^3 - r) = \frac{1}{4}n(n+1)(n^2 + n - 2)$$

$$\Sigma(r^3 - r) = \frac{1}{4}n(n+1)(n-1)(n+2)$$

$$= \frac{1}{4}(n-1)n(n+1)(n+2) \text{ (shown)}$$

b)

Sum to 30 minus sum to 10

$$f(30) = \frac{1}{4} \times 29 \times 30 \times 31 \times 32 = 215760$$

$$f(10) = \frac{1}{4} \times 9 \times 10 \times 11 \times 12 = 2970$$

$$\Sigma(r = 11 \text{ to } 30) = 215760 - 2970$$

212790

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7

a. Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$

b. Hence find the value of $11^2 + 12^2 + 13^2 + \ldots + 20^2$

(5 marks)

a)

$\Sigma(r = n+1 \text{ to } 2n)\, r^2 = \Sigma(r = 1 \text{ to } 2n)\, r^2 - \Sigma(r = 1 \text{ to } n)\, r^2$

Replace n by 2n in the standard result

$$\Sigma(r = 1 \text{ to } 2n)\, r^2 = \frac{1}{6}(2n)(2n+1)(4n+1)$$

$$\Sigma(r = n+1 \text{ to } 2n)\, r^2 = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1)$$

Take out $\frac{1}{6}n(2n+1)$

$$\Sigma(r = n+1 \text{ to } 2n)\, r^2 = \frac{1}{6}n(2n+1)[2(4n+1) - (n+1)]$$

$$\Sigma(r = n+1 \text{ to } 2n)\, r^2 = \frac{1}{6}n(2n+1)(8n+2-n-1)$$

$$\boxed{\Sigma(r = n+1 \text{ to } 2n)\, r^2 = \frac{1}{6}n(2n+1)(7n+1) \text{ (shown)}}$$

b)

From 11 to 20 is n + 1 to 2n with n = 10

$$\text{sum} = \frac{1}{6} \times 10 \times 21 \times 71$$

$$\boxed{2485}$$

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8 Given that $\sum_{r=1}^{n} r^3 = 44\,100$, find the value of n

(4 marks)

$$\frac{1}{4}n^2(n+1)^2 = 44100$$

The left side is a perfect square

$$\left[\frac{1}{2}n(n+1)\right]^2 = 44100$$

n is positive, so take the positive root

$$\frac{1}{2}n(n+1) = 210$$

$$n^2 + n - 420 = 0$$

$$(n+21)(n-20) = 0$$

n must be a positive integer

$$\boxed{n = 20}$$

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9 Given that $\sum_{r=1}^{n} (r^2 + kr) = \frac{1}{6}n(n+1)(2n+7)$ for all positive integers n, where k is a constant,

a. find the value of k

b. hence find the value of $\sum_{r=10}^{25} (r^2 + kr)$

(6 marks)

a)

$$\Sigma(r^2 + kr) = \frac{1}{6}n(n+1)(2n+1) + k \times \frac{1}{2}n(n+1)$$

$$\Sigma(r^2 + kr) = \frac{1}{6}n(n+1)[(2n+1) + 3k]$$

Compare with $\frac{1}{6}n(n+1)(2n+7)$

$$2n + 1 + 3k = 2n + 7$$

$$3k = 6$$

$$\boxed{k = 2}$$

b)

Sum to 25 minus sum to 9

$$f(25) = \frac{1}{6} \times 25 \times 26 \times 57 = 6175$$

$$f(9) = \frac{1}{6} \times 9 \times 10 \times 25 = 375$$

$$\Sigma(r = 10 \text{ to } 25) = 6175 - 375$$

$$\boxed{5800}$$

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10

a. Use the standard results to show that $\sum_{r=1}^{n} (4r^3 - 6r^2 + 4r - 1) = n^4$

b. Hence find the least value of n for which $\sum_{r=1}^{n} (4r^3 - 6r^2 + 4r - 1) > 1\,000\,000$

(7 marks)

a)

$$\Sigma = 4 \times \frac{1}{4}n^2(n+1)^2 - 6 \times \frac{1}{6}n(n+1)(2n+1) + 4 \times \frac{1}{2}n(n+1) - n$$

$$\Sigma = n^2(n+1)^2 - n(n+1)(2n+1) + 2n(n+1) - n$$

Take out $n(n+1)$ from the first three terms

$$\Sigma = n(n+1)[n(n+1) - (2n+1) + 2] - n$$

$$\Sigma = n(n+1)(n^2 - n + 1) - n$$

$$(n+1)(n^2 - n + 1) = n^3 + 1$$

$$\Sigma = n(n^3 + 1) - n$$

$\Sigma = n^4$ (shown)

b)

$n^4 > 1000000$

$n > 1000000^{1/4} = 31.62...$

Check either side

$31^4 = 923521$

$32^4 = 1048576$

$n = 32$

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11 A supermarket builds a display of tins in triangular layers. The top layer is 1 tin, and layer r from the top contains $\frac{1}{2}r(r+1)$ tins.

a. Show that a display with n layers contains $\frac{1}{6}n(n+1)(n+2)$ tins.

b. Find the number of tins in a display with 20 layers.

c. The supermarket has 1000 tins. Find the greatest number of complete layers it can build, and the number of tins left over.

(8 marks)

a)

$$\text{total} = \sum \frac{1}{2}r(r+1) = \frac{1}{2}(\sum r^2 + \sum r)$$

$$\text{total} = \frac{1}{2}\left[\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\right]$$

Take out $\frac{1}{6}n(n+1)$ inside the bracket

$$\text{total} = \frac{1}{2} \times \frac{1}{6}n(n+1)[(2n+1)+3]$$

$$\text{total} = \frac{1}{12}n(n+1)(2n+4)$$

$$\boxed{\text{total} = \frac{1}{6}n(n+1)(n+2) \text{ (shown)}}$$

b)

$$\frac{1}{6} \times 20 \times 21 \times 22 = 1540$$

$$\boxed{1540 \text{ tins}}$$

c)

$$\frac{1}{6}n(n+1)(n+2) \leq 1000$$

Try values of n

17 layers: $\frac{1}{6} \times 17 \times 18 \times 19 = 969$

18 layers: $\frac{1}{6} \times 18 \times 19 \times 20 = 1140$

$1140 > 1000$, so 18 layers is too many

Left over $= 1000 - 969 = 31$

$$\boxed{17 \text{ layers, } 31 \text{ tins left over}}$$

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12

a. Show that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$

b. Hence show that $\sum_{r=n+1}^{3n} r(3r+1) = 2n(13n^2 + 8n + 1)$

c. Given that $\sum_{r=n+1}^{3n} r(3r+1) = 3660$, find the value of n

(8 marks)

a)

$$\Sigma r(3r+1) = 3\Sigma r^2 + \Sigma r$$

$$\Sigma r(3r+1) = 3 \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$\Sigma r(3r+1) = \frac{1}{2}n(n+1)[(2n+1)+1]$$

$$\Sigma r(3r+1) = \frac{1}{2}n(n+1)(2n+2)$$

$$\boxed{\Sigma r(3r+1) = n(n+1)^2 \text{ (shown)}}$$

b)

$f(3n) - f(n)$, where $f(n) = n(n+1)^2$

$$\Sigma(r = n+1 \text{ to } 3n) = 3n(3n+1)^2 - n(n+1)^2$$

$$\Sigma(r = n+1 \text{ to } 3n) = n[3(9n^2 + 6n + 1) - (n^2 + 2n + 1)]$$

$$\Sigma(r = n+1 \text{ to } 3n) = n(27n^2 + 18n + 3 - n^2 - 2n - 1)$$

$$\Sigma(r = n+1 \text{ to } 3n) = n(26n^2 + 16n + 2)$$

$$\boxed{\Sigma(r = n+1 \text{ to } 3n) = 2n(13n^2 + 8n + 1) \text{ (shown)}}$$

c)

$$2n(13n^2 + 8n + 1) = 3660$$

$$26n^3 + 16n^2 + 2n - 3660 = 0$$

Calculator (cubic solver): one real root

$$n = 5$$

Check: $2 \times 5 \times (325 + 40 + 1) = 10 \times 366 = 3660$

$$\boxed{n = 5}$$

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