Vectors: lines and planes

Worked solutions

Further Maths topic practice

This topic online

1 The points A and B have coordinates (2, −1, 3) and (5, 1, −1) respectively. The line l passes through A and B.

a. Find a vector equation for l.

b. Find a cartesian equation for l.

c. Show that the point C(11, 5, −9) lies on l.

(5 marks)

a) Direction = AB = OB - OA

$AB = (5 - 2, 1 + 1, -1 - 3)$

$AB = (3, 2, -4)$

r = point + λ × direction

$$\mathbf{r} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$$

b) Write each component in terms of λ

$x = 2 + 3\lambda$

$y = 2\lambda - 1$

$z = 3 - 4\lambda$

Make λ the subject of each and equate

$$\frac{(x-2)}{3} = \frac{(y+1)}{2} = \frac{(3-z)}{4}$$

c) Find λ from the x-coordinate, then check y and z

$2 + 3\lambda = 11$

$\lambda = 3$

$y = 2(3) - 1 = 5$

$z = 3 - 4(3) = -9$

λ = 3 gives (11, 5, −9), so C lies on L

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2 The line l_1 has equation $\mathbf{r} = (\mathbf{i} + 2\mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

The line l_2 has cartesian equation $\frac{x-3}{4} = \frac{y+1}{-4} = \frac{z}{7}$

Find the acute angle between l_1 and l_2, giving your answer in degrees to one decimal place. (4 marks)

Direction of l_2 comes from the denominators

$\mathbf{d}_1 = (2, -1, 2)$

$\mathbf{d}_2 = (4, -4, 7)$

Scalar product of the directions

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 8 + 4 + 14 = 26$

$|\mathbf{d}_1| = \sqrt{(4 + 1 + 4)} = 3$

$|\mathbf{d}_2| = \sqrt{(16 + 16 + 49)} = 9$

$\cos\theta = \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2)}{(|\mathbf{d}_1||\mathbf{d}_2|)}$

$\cos\theta = \frac{26}{(3 \times 9)}$

$\cos\theta = \frac{26}{27}$

$\theta = 15.64...^\circ$

15.6°

3 The plane Π passes through the point $A(3, -2, 1)$ and is perpendicular to the vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

a. Find a vector equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$.

b. Write down a cartesian equation of Π.

c. Determine whether the point $B(5, 1, 2)$ lies on Π. (4 marks)

a) $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$\mathbf{a} \cdot \mathbf{n} = (3, -2, 1) \cdot (2, -1, 4)$

$\mathbf{a} \cdot \mathbf{n} = 6 + 2 + 4 = 12$

$\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 12$

b) Put $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$2x - y + 4z = 12$

c) Substitute B into the left-hand side

$2(5) - 1 + 4(2) = 17$

$17 \neq 12$

B does not lie on Π

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4 The lines l_1 and l_2 have equations

l_1: $\mathbf{r} = (2\mathbf{i} + \mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

l_2: $\mathbf{r} = (3\mathbf{j} + 4\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

a. Show that l_1 and l_2 are skew.

b. The line l_3 has direction $a\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, where a is a constant. Given that l_3 is perpendicular to l_1, find the value of a.

(6 marks)

a) Not parallel: directions are not multiples

$(1, 2, -1) \neq k(1, 1, 2)$

Equate the components

$2 + \lambda = \mu$

$2\lambda = 3 + \mu$

$1 - \lambda = 4 + 2\mu$

Substitute the first into the second

$2\lambda = 3 + 2 + \lambda$

$\lambda = 5$

$\mu = 7$

Check the z equation

$1 - \lambda = 1 - 5 = -4$

$4 + 2\mu = 4 + 14 = 18$

The z values differ (-4 and 18), so the lines do not meet

Not parallel and do not intersect, so skew

b) Perpendicular means the scalar product is 0

$(a, 2, 3) \cdot (1, 2, -1) = 0$

$a + 4 - 3 = 0$

$a = -1$

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5 The lines l_1 and l_2 have equations

l_1: $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

l_2: $\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

a. Show that l_1 and l_2 intersect, and find the coordinates of their point of intersection.

b. Find the acute angle between l_1 and l_2, giving your answer in degrees to one decimal place. (7 marks)

a) Equate the components

$1 + \lambda = 5 + 2\mu$

$2 - \lambda = 1 + \mu$

$2\lambda - 1 = 2 - \mu$

Add the first two to remove λ

$3 = 6 + 3\mu$

$\mu = -1$

$\lambda = 1 - \mu = 2$

Check in the third equation

$2(2) - 1 = 3$

$2 - (-1) = 3$

Both sides equal, so the lines intersect

Put $\lambda = 2$ into L_1

$r = (1 + 2, 2 - 2, -1 + 4)$

$\boxed{(3, 0, 3)}$

b) Scalar product of the directions

$(1, -1, 2) \cdot (2, 1, -1) = 2 - 1 - 2 = -1$

$|d_1| = \sqrt{(1 + 1 + 4)} = \sqrt{6}$

$|d_2| = \sqrt{(4 + 1 + 1)} = \sqrt{6}$

Use the modulus of the scalar product for the acute angle

$\cos\theta = \frac{1}{(\sqrt{6} \times \sqrt{6})} = \frac{1}{6}$

$\theta = 80.40...^\circ$

$\boxed{80.4^\circ}$

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6 The points $A(1, 0, 2)$, $B(3, 1, 0)$ and $C(0, 2, 1)$ lie in the plane Π.

a. Show that the vector $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b. Find a cartesian equation of Π.

c. Find the exact perpendicular distance from the origin to Π, giving your answer as a simplified surd.

(7 marks)

a) Find two vectors in the plane

$AB = (3 - 1, 1 - 0, 0 - 2) = (2, 1, -2)$

$AC = (0 - 1, 2 - 0, 1 - 2) = (-1, 2, -1)$

Perpendicular means the scalar product is 0

$(3, 4, 5) \cdot (2, 1, -2) = 6 + 4 - 10 = 0$

$(3, 4, 5) \cdot (-1, 2, -1) = -3 + 8 - 5 = 0$

Perpendicular to both AB and AC

b) $(3, 4, 5)$ is a normal; use point A

$3x + 4y + 5z = 3(1) + 4(0) + 5(2)$

$$\boxed{3x + 4y + 5z = 13}$$

c) Distance $= \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{(a^2 + b^2 + c^2)}}$

$\text{distance} = \dfrac{|0 + 0 + 0 - 13|}{\sqrt{(9 + 16 + 25)}}$

$\text{distance} = \dfrac{13}{\sqrt{50}}$

$\text{distance} = \dfrac{13}{(5\sqrt{2})}$

$$\boxed{\frac{13\sqrt{2}}{10}}$$

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7 The plane Π has equation $x - 2y + 2z = 9$

The line l has equation $\mathbf{r} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$.

a. Find the coordinates of the point where l meets Π.

b. Find the acute angle between l and Π, giving your answer in degrees to one decimal place. (7 marks)

a) General point on l

$(1 + 2\lambda, 3 + \lambda, -2 + 3\lambda)$

Substitute into the plane

$$(1 + 2\lambda) - 2(3 + \lambda) + 2(-2 + 3\lambda) = 9$$
$$1 + 2\lambda - 6 - 2\lambda - 4 + 6\lambda = 9$$
$$6\lambda - 9 = 9$$
$$\lambda = 3$$
$$r = (1 + 6, 3 + 3, -2 + 9)$$

$\boxed{(7, 6, 7)}$

b) Angle with the normal, then subtract from 90° : use sin

$$d \cdot n = (2, 1, 3) \cdot (1, -2, 2) = 2 - 2 + 6 = 6$$
$$|d| = \sqrt{(4 + 1 + 9)} = \sqrt{14}$$
$$|n| = \sqrt{(1 + 4 + 4)} = 3$$
$$\sin\theta = \frac{6}{(3\sqrt{14})}$$
$$\sin\theta = \frac{2}{\sqrt{14}}$$
$$\theta = 32.31...^\circ$$

$\boxed{32.3^\circ}$

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8 The plane Π_1 has vector equation

$\mathbf{r} = (2\mathbf{i} + \mathbf{j}) + s(\mathbf{i} + 3\mathbf{k}) + t(2\mathbf{j} - \mathbf{k})$

where s and t are scalar parameters.

a. Show that the vector $6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is a normal to Π_1.

b. Hence find a cartesian equation of Π_1.

The plane Π_2 has equation $x + y + z = 5$

c. Find the acute angle between Π_1 and Π_2, giving your answer in degrees to one decimal place. (7 marks)

a) A normal is perpendicular to both direction vectors

$(6, -1, -2) \cdot (1, 0, 3) = 6 + 0 - 6 = 0$

$(6, -1, -2) \cdot (0, 2, -1) = 0 - 2 + 2 = 0$

$6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is a normal to Π_1

b) Use the point $(2, 1, 0)$

$6x - y - 2z = 6(2) - 1 - 2(0)$

$6x - y - 2z = 11$

c) Angle between planes = angle between normals

$n_1 \cdot n_2 = (6, -1, -2) \cdot (1, 1, 1) = 3$

$|n_1| = \sqrt{(36 + 1 + 4)} = \sqrt{41}$

$|n_2| = \sqrt{3}$

$\cos\theta = \dfrac{3}{(\sqrt{41} \times \sqrt{3})}$

$\cos\theta = \dfrac{3}{\sqrt{123}}$

$\theta = 74.30...^\circ$

74.3°

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9 The plane Π has equation $2x + y - 2z = 4$ and the point P has coordinates $(6, 4, -3)$.

a. Find the perpendicular distance from P to Π.

b. Find the coordinates of the foot of the perpendicular from P to Π.

c. The point Q is the reflection of P in Π. Find the coordinates of Q.

(7 marks)

a) Distance $= \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{(a^2 + b^2 + c^2)}}$

$$\text{distance} = \frac{|2(6) + 4 - 2(-3) - 4|}{\sqrt{(4 + 1 + 4)}}$$

$$\text{distance} = \frac{18}{3}$$

$$\boxed{6}$$

b) Line through P in the direction of the normal

$$r = (6 + 2t, 4 + t, -3 - 2t)$$

Substitute into the plane

$$2(6 + 2t) + (4 + t) - 2(-3 - 2t) = 4$$

$$12 + 4t + 4 + t + 6 + 4t = 4$$

$$22 + 9t = 4$$

$$t = -2$$

$$r = (6 - 4, 4 - 2, -3 + 4)$$

$$\boxed{(2, 2, 1)}$$

c) The foot is the midpoint of PQ, so go twice as far: $t = -4$

$$r = (6 - 8, 4 - 4, -3 + 8)$$

$$\boxed{Q(-2, 0, 5)}$$

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10 The line l has equation $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and the point A has coordinates (6, 4, 5).

The point F lies on l and AF is perpendicular to l.

a. Find the coordinates of F.

b. Hence find the exact shortest distance from A to l.

The point B on l has parameter $\lambda = 4$

c. Find the exact area of triangle ABF.

(8 marks)

a) General point on l

$$F = (1 + 2\lambda, -1 + \lambda, 2 - 2\lambda)$$

$$AF = (2\lambda - 5, \lambda - 5, -3 - 2\lambda)$$

AF is perpendicular to the direction of l

$$2(2\lambda - 5) + (\lambda - 5) - 2(-3 - 2\lambda) = 0$$

$$4\lambda - 10 + \lambda - 5 + 6 + 4\lambda = 0$$

$$9\lambda - 9 = 0$$

$$\lambda = 1$$

$$\boxed{F(3, 0, 0)}$$

b) Shortest distance = $|AF|$

$$AF = (3 - 6, 0 - 4, 0 - 5) = (-3, -4, -5)$$

$$|AF| = \sqrt{(9 + 16 + 25)} = \sqrt{50}$$

$$\boxed{5\sqrt{2}}$$

c) Find B, then BF

$$B = (1 + 8, -1 + 4, 2 - 8) = (9, 3, -6)$$

$$BF = (3 - 9, 0 - 3, 0 + 6) = (-6, -3, 6)$$

$$|BF| = \sqrt{(36 + 9 + 36)} = 9$$

Right angle at F, so area = $\frac{1}{2}$ x base x height

$$\text{area} = \frac{1}{2} \times 5\sqrt{2} \times 9$$

$$\boxed{\frac{45\sqrt{2}}{2}}$$

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11 A drone is launched from a point A on level ground. Relative to a fixed origin, A has position vector $\mathbf{i} + 2\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north and $\mathbf{k}$ is a unit vector vertically upwards. Distances are in metres.

The drone flies in a straight line in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ until it reaches the sloping roof of a barn. The roof is modelled as part of the plane with equation $x + 2y + 2z = 29$

a. Find the position vector of the point where the drone reaches the roof.

b. Find the distance the drone travels.

c. Find the acute angle between the path of the drone and the roof, in degrees to one decimal place.

d. Find the shortest distance from A to the plane of the roof.

e. State one limitation of the model.

(8 marks)

a) Equation of the path

$$r = (1 + 2\lambda, 2 + \lambda, 2\lambda)$$

Substitute into the plane

$$(1 + 2\lambda) + 2(2 + \lambda) + 2(2\lambda) = 29$$
$$1 + 2\lambda + 4 + 2\lambda + 4\lambda = 29$$
$$5 + 8\lambda = 29$$
$$\lambda = 3$$
$$r = (1 + 6, 2 + 3, 6)$$

$$7\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

b) Distance = $\lambda \times$ |direction|

$$|2\mathbf{i} + \mathbf{j} + 2\mathbf{k}| = \sqrt{(4 + 1 + 4)} = 3$$
$$\text{distance} = 3 \times 3$$

$$9 \text{ m}$$

c) Line and plane: use sin with the normal

$$\mathbf{d} \cdot \mathbf{n} = (2, 1, 2) \cdot (1, 2, 2) = 2 + 2 + 4 = 8$$
$$|\mathbf{n}| = \sqrt{(1 + 4 + 4)} = 3$$
$$\sin\theta = \frac{8}{(3 \times 3)} = \frac{8}{9}$$
$$\theta = 62.73...^\circ$$

$$62.7^\circ$$

d) Perpendicular distance from A(1, 2, 0)

$$\text{distance} = \frac{|1 + 2(2) + 2(0) - 29|}{3}$$

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$$\text{distance} = \frac{24}{3}$$

8 m

e) Any sensible limitation

Drone modelled as a particle in an exact straight line (wind would push it off course)

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12 The line l_1 has equation $\mathbf{r} = (\mathbf{i} + 2\mathbf{j}) + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and the point P has coordinates (2, 4, 3).

The plane Π contains l_1 and the point P.

a. Show that $7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is a normal to Π.

b. Find a cartesian equation of Π.

The line l_2 has equation $\mathbf{r} = (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

c. Find the coordinates of the point where l_2 meets Π.

d. Find the acute angle between l_2 and Π, giving your answer in degrees to one decimal place. (9 marks)

a) Two vectors in the plane: the direction of l_1 and AP

$A(1, 2, 0)$ is on l_1 $(\lambda = 0)$

$$AP = (2 - 1, 4 - 2, 3 - 0) = (1, 2, 3)$$

$$(7, 1, -3) \cdot (1, -1, 2) = 7 - 1 - 6 = 0$$

$$(7, 1, -3) \cdot (1, 2, 3) = 7 + 2 - 9 = 0$$

Perpendicular to both, so $7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is a normal

b) Use the point $A(1, 2, 0)$

$$7x + y - 3z = 7(1) + 2 - 3(0)$$

$$\boxed{7x + y - 3z = 9}$$

c) General point on l_2, substitute into Π

$$7(1 + \mu) + (-2 + \mu) - 3(2 + \mu) = 9$$

$$7 + 7\mu - 2 + \mu - 6 - 3\mu = 9$$

$$5\mu - 1 = 9$$

$$\mu = 2$$

$$r = (1 + 2, -2 + 2, 2 + 2)$$

$$\boxed{(3, 0, 4)}$$

d) Line and plane: $\sin\theta = \dfrac{|\mathbf{d} \cdot \mathbf{n}|}{(|\mathbf{d}||\mathbf{n}|)}$

$$\mathbf{d} \cdot \mathbf{n} = (1, 1, 1) \cdot (7, 1, -3) = 5$$

$$|\mathbf{d}| = \sqrt{3}$$

$$|\mathbf{n}| = \sqrt{(49 + 1 + 9)} = \sqrt{59}$$

$$\sin\theta = \frac{5}{(\sqrt{3} \times \sqrt{59})} = \frac{5}{\sqrt{177}}$$

$$\theta = 22.07...^\circ$$

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22.1

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