Volumes of revolution

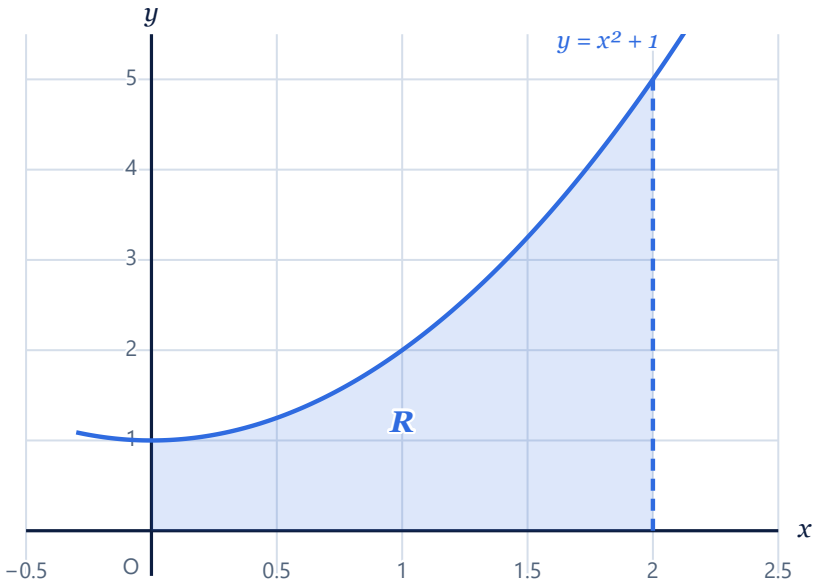
Worked solutions

Further Maths topic practice

This topic online

1 The diagram shows the region R bounded by the curve with equation $y = x^2 + 1$, the x-axis, the y-axis and the line $x = 2$

The region R is rotated through 2π radians about the x-axis. Use calculus to find the exact volume of the solid generated.



(4 marks)

About the x-axis: $V = \pi\int y^2\,dx$

$$V = \pi\int (x^2+1)^2\,dx$$

$$V = \pi\int (x^4+2x^2+1)\,dx$$

Integrate, limits 0 to 2

$$V = \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]$$

$$V = \pi\left(\frac{32}{5} + \frac{16}{3} + 2\right) - 0$$

$$V = \pi\left(\frac{96}{15} + \frac{80}{15} + \frac{30}{15}\right)$$

$$\boxed{V = \frac{206\pi}{15}}$$

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2 The region bounded by the curve with equation $y = 2\sqrt{x}$, the x-axis and the lines $x = 1$ and $x = 4$ is rotated through 360° about the x-axis.

Find the exact volume of the solid formed. (3 marks)

$V = \pi \int y^2 \, dx$

$y^2 = (2\sqrt{x})^2 = 4x$

$V = \pi \int 4x \, dx$

Limits 1 to 4

$V = \pi [2x^2]$

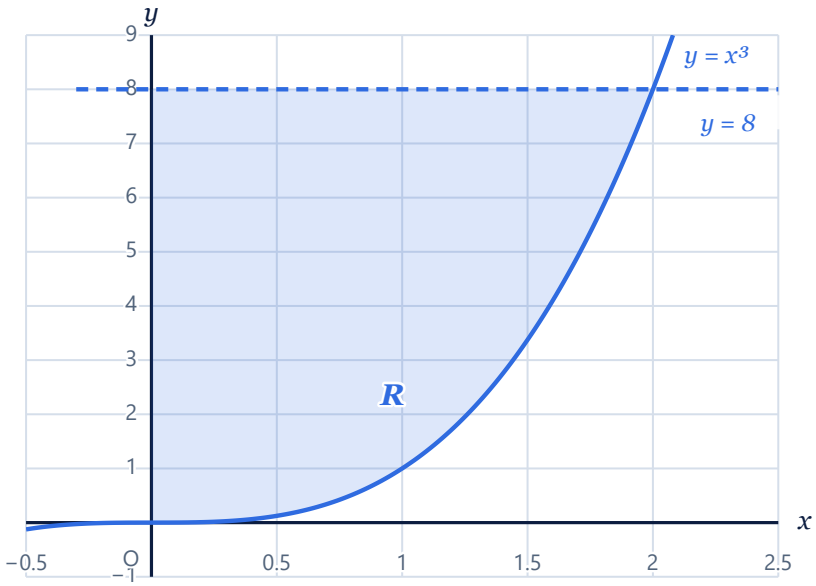
$V = \pi (2 \times 16 - 2 \times 1)$

$\boxed{V = 30\pi}$

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3 The diagram shows the region R bounded by the curve with equation $y = x^3$, the y-axis and the line $y = 8$

The region R is rotated through 2π radians about the y-axis. Find the exact volume of the solid generated.



(5 marks)

About the y-axis: $V = \pi\int x^2\,dy$

$x = y^{1/3}$

$x^2 = y^{2/3}$

$V = \pi\int y^{2/3}\,dy$

Limits in y: 0 to 8

$V = \pi\left[\frac{3}{5}y^{5/3}\right]$

$V = \frac{3\pi}{5} \times 8^{5/3}$

$8^{5/3} = 2^5 = 32$

$V = \frac{3\pi}{5} \times 32$

$$\boxed{V = \frac{96\pi}{5}}$$

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4 The curve C has equation $y = \frac{3}{x}$, $x > 0$

The region bounded by C, the x-axis and the lines $x = 1$ and $x = 3$ is rotated through 360° about the x-axis.

Find the volume of the solid formed, giving your answer as an exact multiple of π (4 marks)

$$y^2 = \frac{9}{x^2} = 9x^{-2}$$

$$V = \pi \int 9x^{-2}\, dx$$

Limits 1 to 3

$$V = \pi [-9x^{-1}]$$

$$V = \pi \left[\left(-\frac{9}{3}\right) - \left(-\frac{9}{1}\right)\right]$$

$$V = \pi(-3 + 9)$$

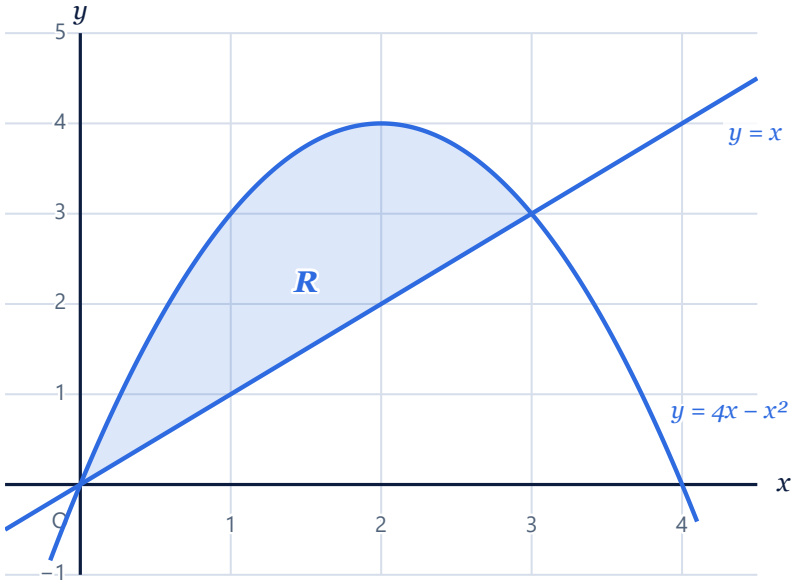
$$\boxed{V = 6\pi}$$

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5 The diagram shows the curve with equation $y = 4x - x^2$ and the line with equation $y = x$. The finite region R lies between the curve and the line.

a. Find the x-coordinates of the points where the line meets the curve.

b. The region R is rotated through 2π radians about the x-axis. Find the exact volume of the solid generated.



(6 marks)

a)

$4x - x^2 = x$

$x^2 - 3x = 0$

$x(x - 3) = 0$

$\boxed{x = 0 \text{ and } x = 3}$

b)

Outer radius = curve, inner radius = line

$$V = \pi\int[(4x - x^2)^2 - x^2]\,dx$$

$(4x - x^2)^2 = 16x^2 - 8x^3 + x^4$

$$V = \pi\int(15x^2 - 8x^3 + x^4)\,dx$$

Limits 0 to 3

$$V = \pi\left[5x^3 - 2x^4 + \frac{x^5}{5}\right]$$

$$V = \pi\left(5 \times 27 - 2 \times 81 + \frac{243}{5}\right)$$

$$V = \pi(135 - 162 + 48.6)$$

$$V = 21.6\pi$$

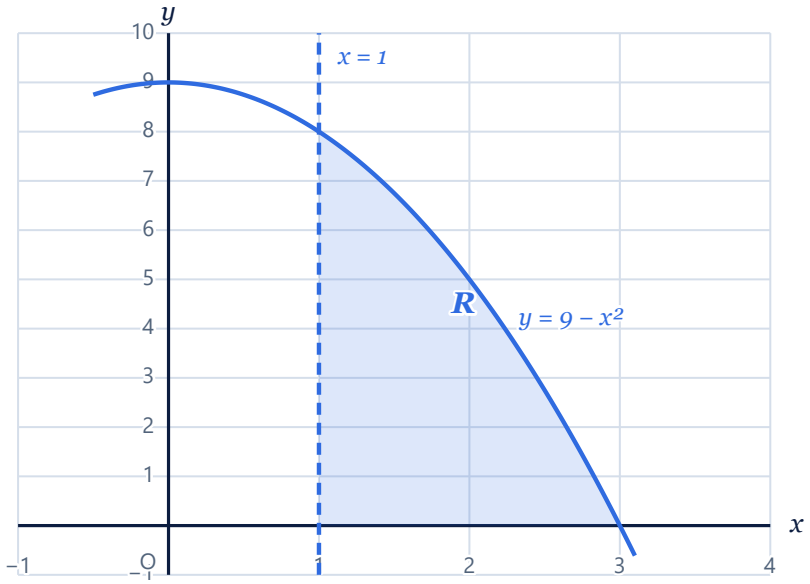
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$$V = \frac{108\pi}{5}$$

6 The diagram shows the region R bounded by the curve with equation $y = 9 - x^2$, the x-axis and the line $x = 1$

The region R is rotated through 360° about the y-axis to form a solid ring.

Find the exact volume of the ring.



(6 marks)

Where $x = 1$ meets the curve

$y = 9 - 1 = 8$

Solid from the curve, y from 0 to 8

$x^2 = 9 - y$

$V_1 = \pi\int(9 - y)\,dy$

Limits 0 to 8

$V_1 = \pi\left[9y - \frac{y^2}{2}\right]$

$V_1 = \pi(72 - 32) = 40\pi$

Subtract the cylinder of radius 1, height 8

$V_2 = \pi \times 1^2 \times 8 = 8\pi$

$V = 40\pi - 8\pi$

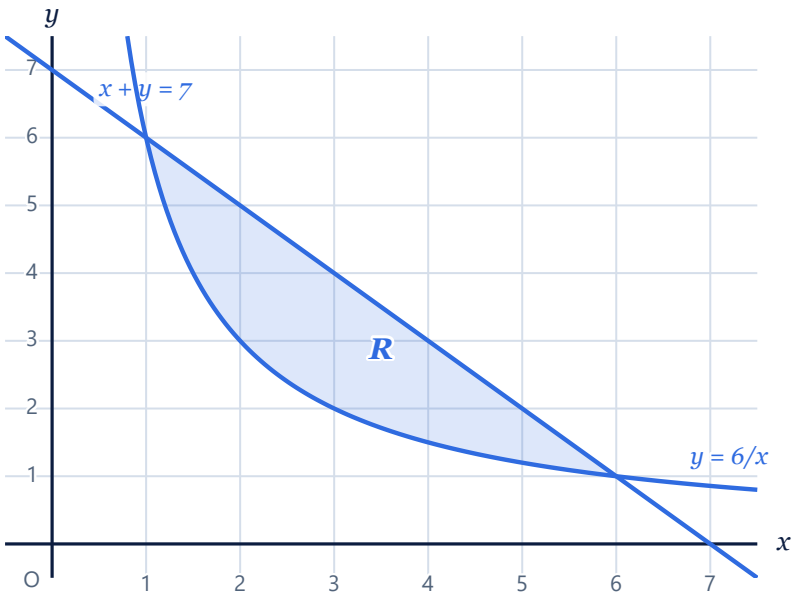
$$V = 32\pi$$

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7 The diagram shows the curve with equation $y = \frac{6}{x}$, $x > 0$, and the line with equation $x + y = 7$

a. Find the x-coordinates of the points where the line meets the curve.

b. The finite region R between the line and the curve is rotated through 2π radians about the x-axis. Use algebraic integration to find the exact volume of the solid generated.



(7 marks)

a)

$$7 - x = \frac{6}{x}$$

$$7x - x^2 = 6$$

$$x^2 - 7x + 6 = 0$$

$$(x - 1)(x - 6) = 0$$

$$\boxed{x = 1 \text{ and } x = 6}$$

b)

Outer radius = line, inner radius = curve

$$V = \pi\int\left[(7 - x)^2 - \frac{36}{x^2}\right] dx$$

Line part, limits 1 to 6

$$\pi\int(49 - 14x + x^2)\,dx = \pi\left[49x - 7x^2 + \frac{x^3}{3}\right]$$

$$= \pi\left[(294 - 252 + 72) - (49 - 7 + \frac{1}{3})\right]$$

$$= \pi\left(114 - \frac{127}{3}\right)$$

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$$= \frac{215\pi}{3}$$

Curve part, limits 1 to 6

$$\pi\int 36x^{-2}\,dx = \pi\left[-\frac{36}{x}\right]$$

$$= \pi(-6 + 36) = 30\pi$$

$$V = \frac{215\pi}{3} - 30\pi$$

$$\boxed{V = \frac{125\pi}{3}}$$

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8 The region R is bounded by the curve with equation $y = \sqrt{x}$, the x-axis and the line $x = 4$

a. Find the exact volume of the solid formed when R is rotated through 2π radians about the x-axis.

b. Find the exact volume of the solid formed when R is rotated through 2π radians about the y-axis. (7 marks)

a)

$$y^2 = x$$

$$V = \pi \int x \, dx$$

Limits 0 to 4

$$V = \pi \left[\frac{x^2}{2}\right]$$

$$V = \pi \times \frac{16}{2}$$

$$\boxed{V = 8\pi}$$

b)

y runs from 0 to 2; the region lies between $x = y^2$ and $x = 4$

So the solid is a cylinder with the curve's solid removed

$$\text{cylinder} = \pi \times 4^2 \times 2 = 32\pi$$

$$x = y^2, \text{ so } x^2 = y^4$$

$$\pi \int y^4 \, dy = \pi \left[\frac{y^5}{5}\right]$$

Limits 0 to 2

$$\pi \int y^4 \, dy = \frac{32\pi}{5}$$

$$V = 32\pi - \frac{32\pi}{5}$$

$$\boxed{V = \frac{128\pi}{5}}$$

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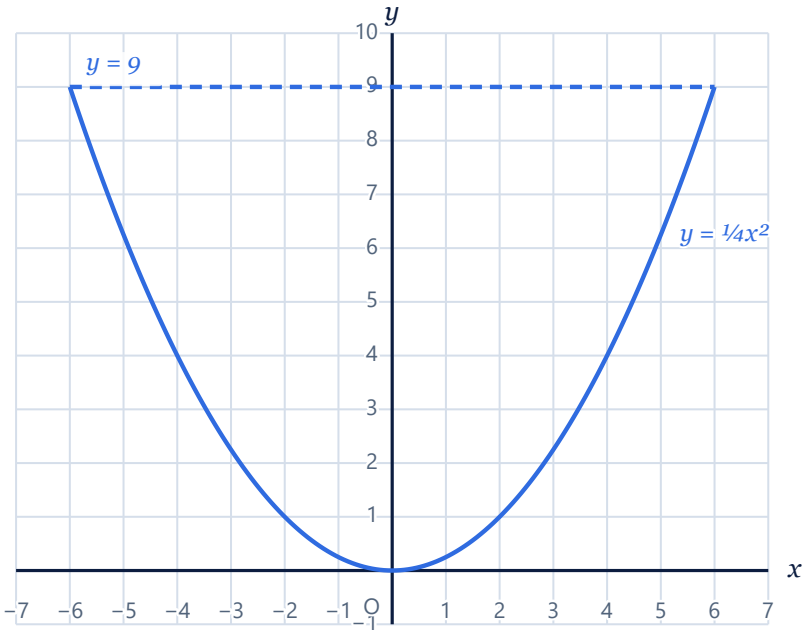
9 The inside of a bowl is modelled by rotating the curve with equation $y = \frac{1}{4}x^2$, for $0 \le y \le 9$, through 360° about the y-axis. The units are centimetres and the y-axis is vertical.

a. Show that, according to the model, the volume of water in the bowl when the water has depth h cm is $2\pi h^2$ cm^3

b. Find the volume of water the bowl holds when it is full.

c. Kofi pours 200 cm^3 of soup into the bowl. Find the depth of the soup.

d. Give one reason why the model may not give the true capacity of the bowl.



(7 marks)

a)

About the y-axis: $V = \pi\int x^2\,dy$

$x^2 = 4y$

$V = \pi\int 4y\,dy$

Limits 0 to h

$V = \pi[2y^2]$

$\boxed{V = 2\pi h^2 \text{ (shown)}}$

b)

$V = 2\pi \times 9^2$

$V = 162\pi$

$\boxed{V = 509 \text{ cm}^3 \text{ (3 s.f.)}}$

c)

$2\pi h^2 = 200$

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$$h^2 = \frac{100}{\pi}$$

$$h = \sqrt{31.83...}$$

$h = 5.64$ cm (3 s.f.)

d)

A real bowl is not exactly this curve (for example a flat base)

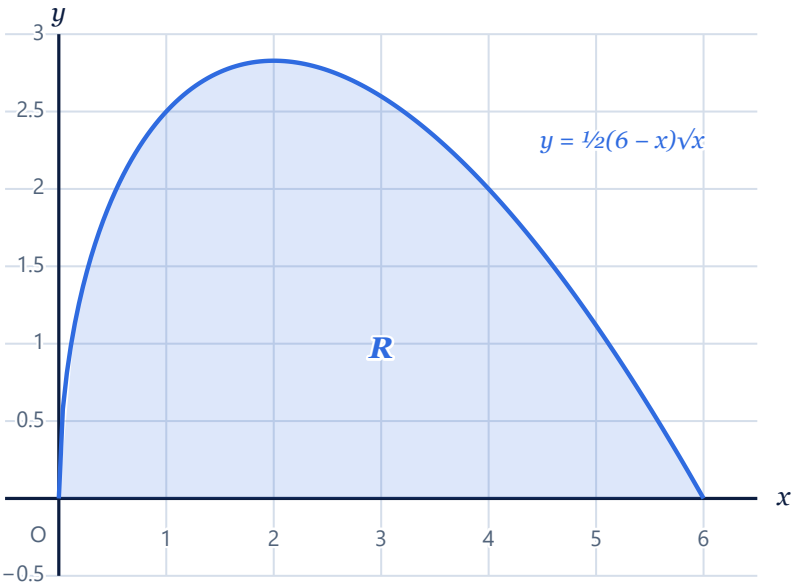
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10 A wooden door knob is modelled by rotating the region R, bounded by the curve with equation $y = \frac{1}{2}(6 - x)\sqrt{x}$ and the x-axis, through 2π radians about the x-axis. The units are centimetres.

a. Use algebraic integration to show that the volume of the knob, according to the model, is 27π cm^3

b. The wood has a density of 0.7 grams per cm^3. Find the mass of the knob according to the model.

c. A knob made this way has a mass of 62 g. Comment on the model.



(7 marks)

a)

$$y^2 = \frac{1}{4}x(6 - x)^2$$

$$y^2 = \frac{1}{4}(36x - 12x^2 + x^3)$$

$$V = \frac{\pi}{4}\int (36x - 12x^2 + x^3)\,dx$$

The curve meets the x-axis at $x = 0$ and $x = 6$

$$V = \frac{\pi}{4}\left[18x^2 - 4x^3 + \frac{x^4}{4}\right]$$

$$V = \frac{\pi}{4}(648 - 864 + 324)$$

$$V = \frac{\pi}{4} \times 108$$

$$\boxed{V = 27\pi \text{ cm}^3 \text{ (shown)}}$$

b)

$$\text{mass} = 0.7 \times 27\pi$$

$$\text{mass} = 59.37...$$

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59.4 g

c)

$(62 - 59.4) \div 59.4 = 0.044$

about 4% heavier than the model

Close, so the model is reasonable; it slightly underestimates the mass

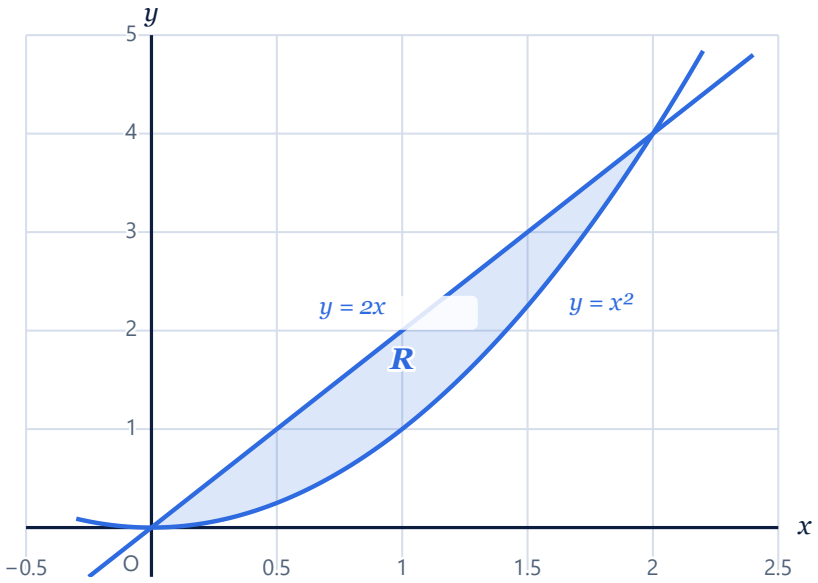
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11 The diagram shows the finite region R bounded by the curve with equation $y = x^2$ and the line with equation $y = 2x$

a. Find the coordinates of the points where the line meets the curve.

b. Find the exact volume of the solid formed when R is rotated through 2π radians about the x-axis.

c. Find the exact volume of the solid formed when R is rotated through 2π radians about the y-axis.



(9 marks)

a)

$$x^2 = 2x$$

$$x(x - 2) = 0$$

$$\boxed{(0, 0) \text{ and } (2, 4)}$$

b)

For $0 < x < 2$ the line is above the curve: outer radius $2x$, inner radius x^2

$$V = \pi\int[(2x)^2 - (x^2)^2]\,dx$$

$$V = \pi\int(4x^2 - x^4)\,dx$$

Limits 0 to 2

$$V = \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]$$

$$V = \pi\left(\frac{32}{3} - \frac{32}{5}\right)$$

$$V = \pi\left(\frac{160}{15} - \frac{96}{15}\right)$$

$$\boxed{V = \frac{64\pi}{15}}$$

c)

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In terms of y: curve $x = \sqrt{y}$ (outside), line $x = \frac{y}{2}$ (inside), y from 0 to 4

$$\text{curve solid} = \pi \int y \, dy = \pi \left[\frac{y^2}{2} \right]$$

$$\text{curve solid} = 8\pi$$

The line gives a cone: radius 2, height 4

$$\text{cone} = \frac{1}{3} \pi \times 2^2 \times 4 = \frac{16\pi}{3}$$

$$V = 8\pi - \frac{16\pi}{3}$$

$$\boxed{V = \frac{8\pi}{3}}$$

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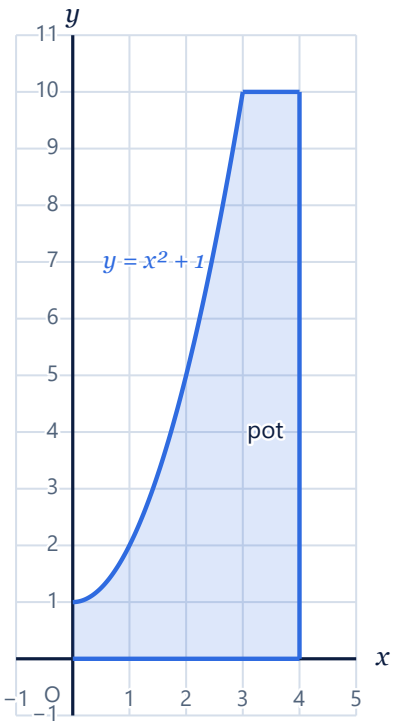
12 A ceramic plant pot is modelled as a solid cylinder of radius 4 cm and height 10 cm with its inside removed. The inside surface is modelled by rotating the curve with equation $y = x^2 + 1$, for $1 \leq y \leq 10$, through 360° about the y-axis. The base of the pot lies on $y = 0$, and the diagram shows the right-hand half of a vertical cross-section through the centre.

a. Show that the volume of the inside of the pot is $\frac{81}{2}\pi$ cm^3

b. Find the exact volume of ceramic in the pot.

c. The ceramic has a density of 2.3 g per cm^3. Find the mass of the pot.

d. The pot is filled with compost to 2 cm below the rim. Find the volume of compost needed.



(9 marks)

a)

About the y-axis: $V = \pi\int x^2\,dy$

$x^2 = y - 1$

$V = \pi\int (y - 1)\,dy$

Limits 1 to 10

$V = \pi\left[\frac{y^2}{2} - y\right]$

$V = \pi\left[(50 - 10) - \left(\frac{1}{2} - 1\right)\right]$

$V = \pi\left(40 + \frac{1}{2}\right)$

$$V = \frac{81\pi}{2}\ \text{cm}^3\ \text{(shown)}$$

b)

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$$\text{cylinder} = \pi \times 4^2 \times 10 = 160\pi$$

$$\text{ceramic} = 160\pi - \frac{81\pi}{2}$$

$$\boxed{\text{ceramic} = \frac{239\pi}{2}\ \text{cm}^3}$$

c)

$$\frac{239\pi}{2} = 375.4\ldots\ \text{cm}^3$$

$$\text{mass} = 2.3 \times 375.4\ldots$$

$$\boxed{\text{mass} = 863\ \text{g (3 s.f.)}}$$

d)

Rim at $y = 10$, so compost goes up to $y = 8$

$$V = \pi\left[\frac{y^2}{2} - y\right]$$

Limits 1 to 8

$$V = \pi\left[(32 - 8) - \left(\frac{1}{2} - 1\right)\right]$$

$$V = \frac{49\pi}{2}$$

$$\boxed{V = 77.0\ \text{cm}^3\ \text{(3 s.f.)}}$$

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