Areas and tangents in polar coordinates

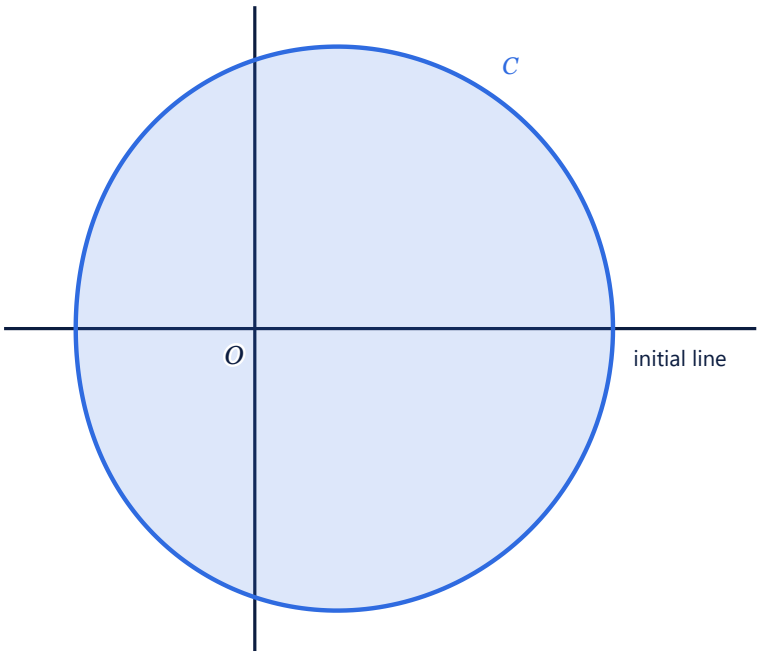
Worked solutions

Further Maths topic practice

This topic online

1 The diagram shows the curve C with polar equation $r = 3 + \cos\theta$, $0 \leq \theta < 2\pi$

Find the exact area of the region enclosed by C.



(4 marks)

$$A = \frac{1}{2}\int r^2 \, d\theta$$

$$r^2 = 9 + 6\cos\theta + \cos^2\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$r^2 = \frac{19}{2} + 6\cos\theta + \frac{1}{2}\cos 2\theta$$

Limits $\theta = 0$ to $\theta = 2\pi$

$$A = \frac{1}{2}\left[\frac{19}{2}\theta + 6\sin\theta + \frac{1}{4}\sin 2\theta\right]$$

$$A = \frac{1}{2}\left(\frac{19}{2} \times 2\pi + 0 + 0\right)$$

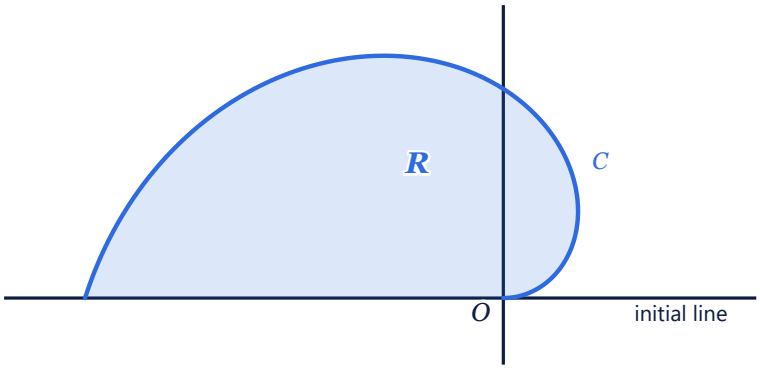
$$\boxed{A = \frac{19\pi}{2}}$$

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2 The curve C has polar equation $r = 2\theta$, $0 \leq \theta \leq \pi$

The region R, shown shaded in the diagram, is bounded by C and the half-line $\theta = \pi$

Find the exact area of R.



(3 marks)

$$A = \frac{1}{2}\int r^2 \, d\theta$$

$$A = \frac{1}{2}\int 4\theta^2 \, d\theta$$

Limits $\theta = 0$ to $\theta = \pi$

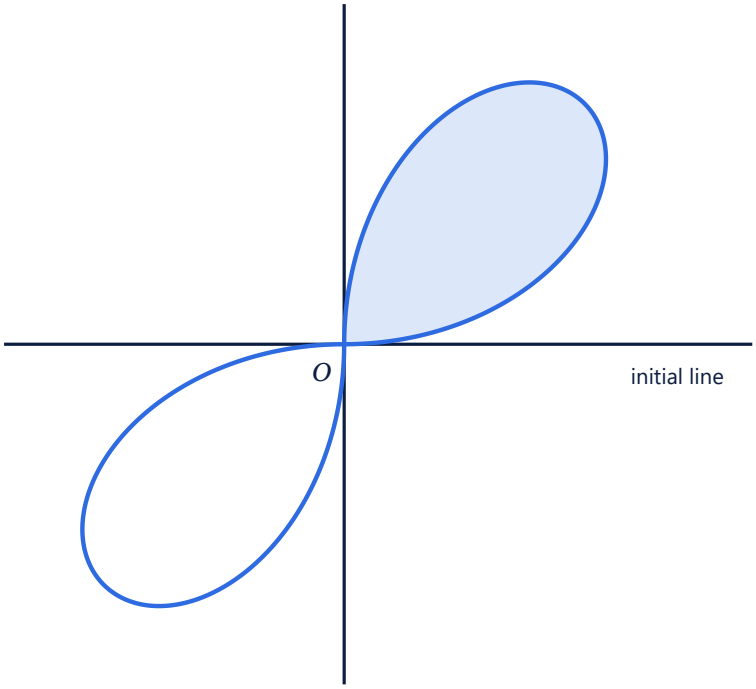
$$A = \left[\frac{2\theta^3}{3}\right]$$

$$\boxed{A = \frac{2\pi^3}{3}}$$

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3 The diagram shows the curve with polar equation $r = 4 \sin 2\theta$, for values of θ where $r \geq 0$

Find the exact area of the shaded loop, which lies in $0 \leq \theta \leq \frac{\pi}{2}$



(4 marks)

$$A = \frac{1}{2}\int 16\sin^2 2\theta \, d\theta$$

$$\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$$

$$A = 4\int (1 - \cos 4\theta) \, d\theta$$

Limits $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = 4\left[\theta - \frac{1}{4}\sin 4\theta\right]$$

$$A = 4\left(\frac{\pi}{2} - 0\right)$$

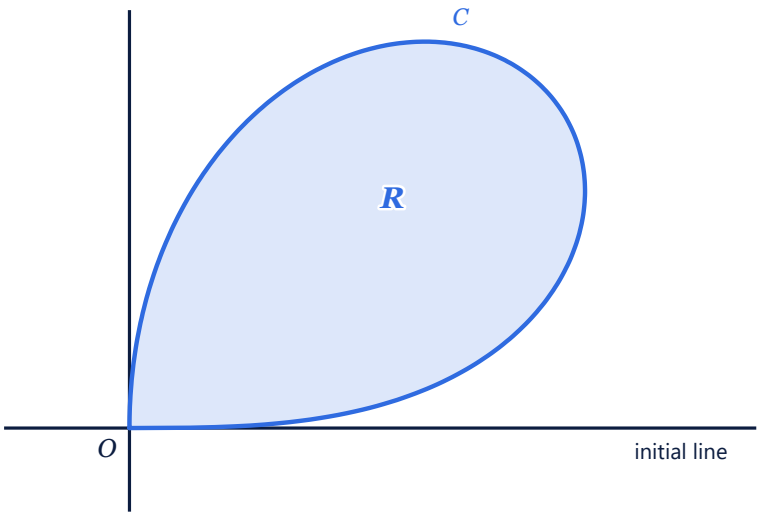
$$\boxed{A = 2\pi}$$

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4 The curve C has polar equation $r = \sqrt{\theta}\cos\theta$, $0 \leq \theta \leq \frac{\pi}{2}$

The region R, shown shaded in the diagram, is enclosed by C.

Use algebraic integration to show that the area of R is $\frac{\pi^2}{32} - \frac{1}{8}$



(6 marks)

$$A = \frac{1}{2}\int \theta\cos^2\theta \, d\theta$$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$A = \frac{1}{4}\int (\theta + \theta\cos 2\theta) \, d\theta$$

Integrate $\theta\cos 2\theta$ by parts: $u = \theta$, $\frac{dv}{d\theta} = \cos 2\theta$

$$\int \theta\cos 2\theta \, d\theta = \frac{1}{2}\theta\sin 2\theta - \int \frac{1}{2}\sin 2\theta \, d\theta$$

$$\int \theta\cos 2\theta \, d\theta = \frac{1}{2}\theta\sin 2\theta + \frac{1}{4}\cos 2\theta$$

$$A = \frac{1}{4}\left[\frac{1}{2}\theta^2 + \frac{1}{2}\theta\sin 2\theta + \frac{1}{4}\cos 2\theta\right]$$

Limits $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$A = \frac{1}{4}\left(\left(\frac{\pi^2}{8} + 0 - \frac{1}{4}\right) - \left(0 + 0 + \frac{1}{4}\right)\right)$$

$$A = \frac{1}{4}\left(\frac{\pi^2}{8} - \frac{1}{2}\right)$$

$$\boxed{A = \frac{\pi^2}{32} - \frac{1}{8} \text{ (shown)}}$$

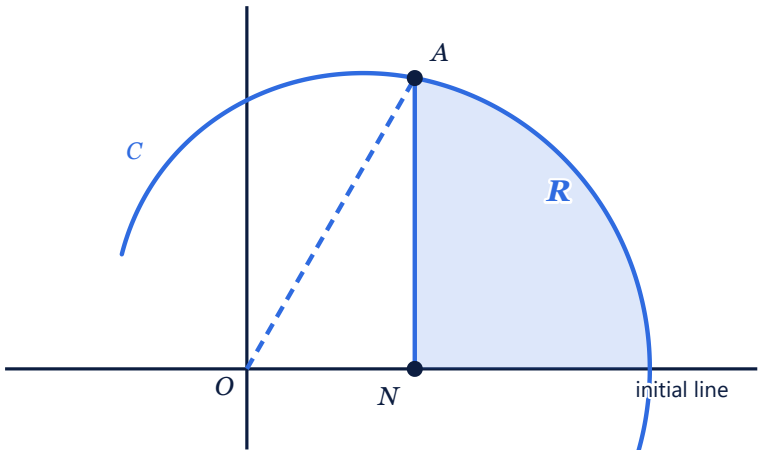
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5 The diagram shows part of the curve C with polar equation $r = 2 + \cos\theta$, $0 \leq \theta < 2\pi$

The point A lies on C where $\theta = \frac{\pi}{3}$. The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded, is bounded by C, the initial line and the line segment AN.

Find the exact area of R, giving your answer in the form $p\pi + q\sqrt{3}$, where p and q are rational numbers.



(8 marks)

R = (polar area from $\theta = 0$ to $\frac{\pi}{3}$) − (triangle OAN)

$r = 2 + \cos\frac{\pi}{3} = \frac{5}{2}$ at A

$ON = \frac{5}{2}\cos\frac{\pi}{3} = \frac{5}{4}$

$AN = \frac{5}{2}\sin\frac{\pi}{3} = \frac{5\sqrt{3}}{4}$

triangle OAN $= \frac{1}{2} \times \frac{5}{4} \times \frac{5\sqrt{3}}{4} = \frac{25\sqrt{3}}{32}$

$r^2 = 4 + 4\cos\theta + \cos^2\theta = \frac{9}{2} + 4\cos\theta + \frac{1}{2}\cos 2\theta$

polar area $= \frac{1}{2}\int\left(\frac{9}{2} + 4\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$

polar area $= \frac{1}{2}\left[\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta\right]$

Limits $\theta = 0$ to $\theta = \frac{\pi}{3}$

polar area $= \frac{1}{2}\left(\frac{3\pi}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{8}\right)$

polar area $= \frac{3\pi}{4} + \frac{17\sqrt{3}}{16}$

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$$R = \frac{3\pi}{4} + \frac{34\sqrt{3}}{32} - \frac{25\sqrt{3}}{32}$$

$$\boxed{R = \frac{3\pi}{4} + \frac{9\sqrt{3}}{32}}$$

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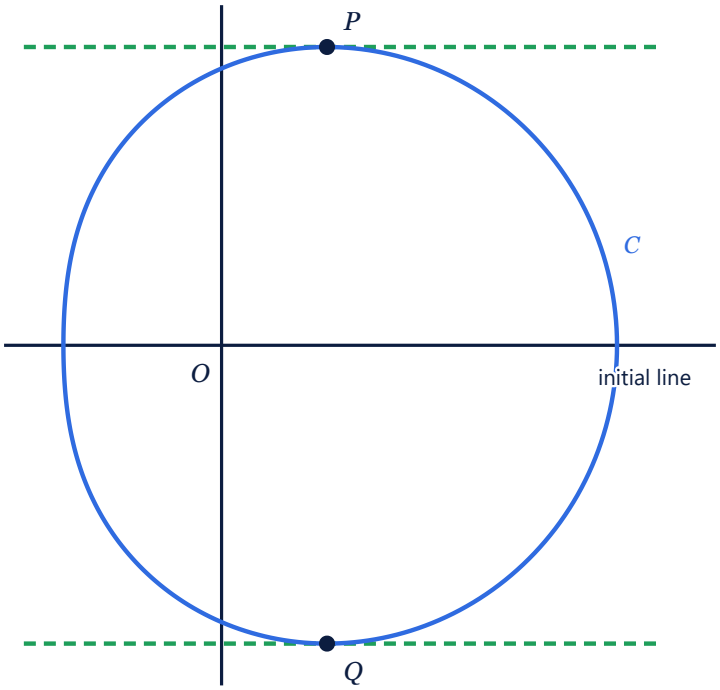
6 The curve C has polar equation $r = 7 + 3\cos\theta$, $0 \leq \theta < 2\pi$

The tangents to C at the points P and Q are parallel to the initial line.

a. Use calculus to show that at P and Q, $\cos\theta = \frac{1}{3}$

b. Find the exact value of r at P, and an equation of the tangent at P in the form $r\sin\theta = k$, where P lies above the initial line.

c. Find the exact distance between the two tangents.



(7 marks)

a)

Parallel to the initial line: $\frac{dy}{d\theta} = 0$ where $y = r\sin\theta$

$$y = (7 + 3\cos\theta)\sin\theta$$

$$y = 7\sin\theta + \frac{3}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = 7\cos\theta + 3\cos 2\theta$$

$$\frac{dy}{d\theta} = 7\cos\theta + 6\cos^2\theta - 3$$

$$6\cos^2\theta + 7\cos\theta - 3 = 0$$

$$(3\cos\theta - 1)(2\cos\theta + 3) = 0$$

$\cos\theta = -\frac{3}{2}$ has no solutions

$$\boxed{\cos\theta = \frac{1}{3} \text{ (shown)}}$$

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b)

$$r = 7 + 3 \times \frac{1}{3} = 8$$

$$\sin\theta = \sqrt{\left(1 - \frac{1}{9}\right)} = \frac{2\sqrt{2}}{3}$$

$$y = r\sin\theta = 8 \times \frac{2\sqrt{2}}{3} = \frac{16\sqrt{2}}{3}$$

$r = 8$, tangent $r\sin\theta = \frac{16\sqrt{2}}{3}$

c)

The curve is symmetrical about the initial line, so the other tangent is $y = -\frac{16\sqrt{2}}{3}$

distance $= \frac{32\sqrt{2}}{3}$

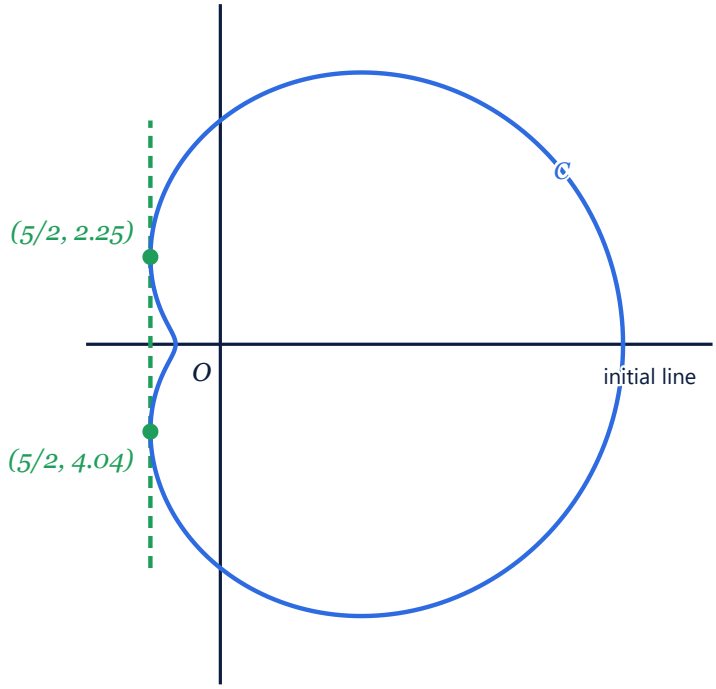
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7 The curve C has polar equation $r = 5 + 4\cos\theta$, $0 \leq \theta < 2\pi$

a. Use calculus to find the polar coordinates of the points on C where the tangent is perpendicular to the initial line, giving values of θ to 3 significant figures where appropriate.

b. Find a cartesian equation of the tangent at the point in part (a) with $0 < \theta < \pi$

c. Explain how your answers show that C has a dimple where it crosses the half-line $\theta = \pi$



(7 marks)

a)

Perpendicular to the initial line: $\frac{dx}{d\theta} = 0$ where $x = r\cos\theta$

$$x = 5\cos\theta + 4\cos^2\theta$$

$$\frac{dx}{d\theta} = -5\sin\theta - 8\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = -\sin\theta(5 + 8\cos\theta)$$

$$\sin\theta = 0: \theta = 0 \text{ or } \pi$$

$$\cos\theta = -\frac{5}{8}: \theta = 2.2459... \text{ or } 4.0372...$$

$$\theta = 0: r = 9$$

$$\theta = \pi: r = 1$$

$$\cos\theta = -\frac{5}{8}: r = 5 - \frac{5}{2} = \frac{5}{2}$$

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$(9, 0), (1, \pi), (\frac{5}{2}, 2.25), (\frac{5}{2}, 4.04)$

b)

$x = r\cos\theta = \frac{5}{2} \times (-\frac{5}{8})$

$x = -\frac{25}{16}$

c)

At $\theta = \pi$: $x = -1$

$-\frac{25}{16} < -1$, so the curve goes further left above and below the initial line

The curve bends inwards at $\theta = \pi$, so there is a dimple

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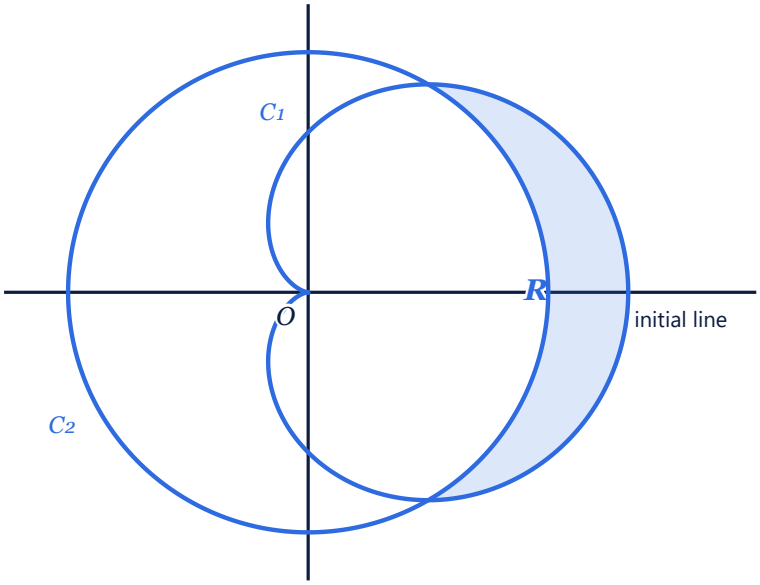
8 The diagram shows the curves C_1 and C_2 with polar equations

$$C_1: r = 2(1 + \cos\theta), \quad 0 \le \theta < 2\pi \qquad C_2: r = 3, \quad 0 \le \theta < 2\pi$$

a. Find the polar coordinates of the points where C_1 and C_2 meet.

The region R, shown shaded, lies inside C_1 and outside C_2.

b. Use algebraic integration to show that the area of R is $\frac{9}{2}\sqrt{3} - \pi$



(9 marks)

a)

$2(1 + \cos\theta) = 3$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$\left(3, \frac{\pi}{3}\right)$ and $\left(3, \frac{5\pi}{3}\right)$

b)

$A = \frac{1}{2}\int (r_1^2 - r_2^2)\, d\theta$

Symmetrical about the initial line: use 2 × (0 to $\frac{\pi}{3}$)

$A = \int (4(1 + \cos\theta)^2 - 9)\, d\theta$

$4(1 + \cos\theta)^2 = 4 + 8\cos\theta + 4\cos^2\theta$

$4\cos^2\theta = 2 + 2\cos 2\theta$

$A = \int (-3 + 8\cos\theta + 2\cos 2\theta)\, d\theta$

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$$A = [-3\theta + 8\sin\theta + \sin 2\theta]$$

Limits $\theta = 0$ to $\theta = \frac{\pi}{3}$

$$A = -\pi + 8 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$A = -\pi + 4\sqrt{3} + \frac{\sqrt{3}}{2}$$

$$\boxed{A = \frac{9}{2}\sqrt{3} - \pi \text{ (shown)}}$$

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9 The diagram shows the curves C_1 and C_2 with polar equations

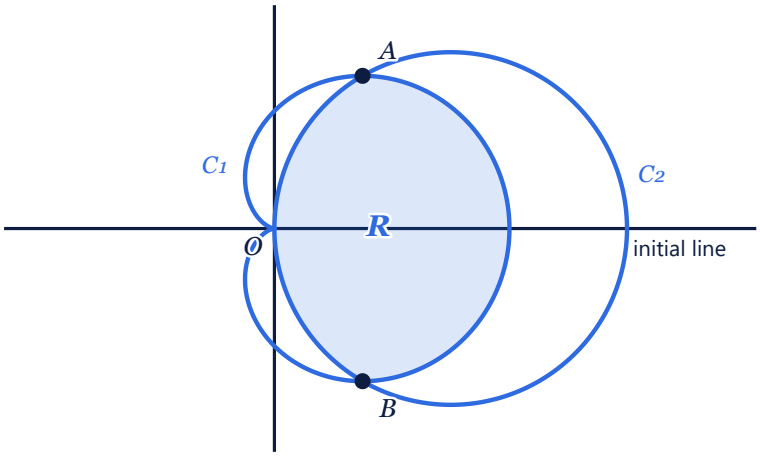
$$C_1: r = 1 + \cos\theta, \quad -\pi < \theta \leq \pi \qquad C_2: r = 3\cos\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

The curves meet at the pole and at the points A and B.

a. Find the polar coordinates of A and B.

The region R, shown shaded, lies inside both curves.

b. Find the exact area of R.



(9 marks)

a)

$$1 + \cos\theta = 3\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

$$r = 3 \times \frac{1}{2} = \frac{3}{2}$$

$$A\left(\frac{3}{2}, \frac{\pi}{3}\right), B\left(\frac{3}{2}, -\frac{\pi}{3}\right)$$

b)

$0 \leq \theta \leq \frac{\pi}{3}$: R is bounded by C_1

$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$: R is bounded by C_2

Double the area above the initial line

$$(1 + \cos\theta)^2 = \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta$$

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$$\int (1+\cos\theta)^2\,d\theta = \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right] = \frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}$$

Limits 0 to $\frac{\pi}{3}$

$$\int 9\cos^2\theta\,d\theta = \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] = \frac{9}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

Limits $\frac{\pi}{3}$ to $\frac{\pi}{2}$

$$\int 9\cos^2\theta\,d\theta = \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}$$

$$A = 2 \times \frac{1}{2}\left(\left(\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right) + \left(\frac{3\pi}{4} - \frac{9\sqrt{3}}{8}\right)\right)$$

$$\boxed{A = \frac{5\pi}{4}}$$

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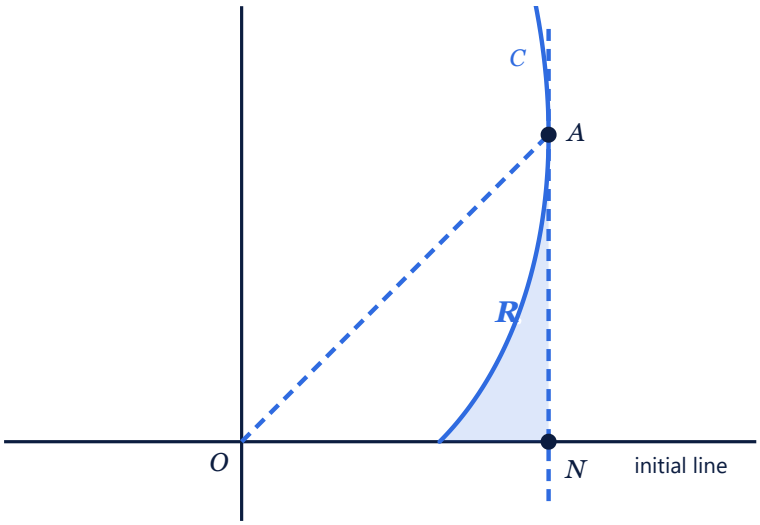
10 The curve C has polar equation $r = e^{\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$

The tangent to C at the point A is perpendicular to the initial line.

a. Use calculus to show that A has polar coordinates $(e^{\pi/4}, \frac{\pi}{4})$

The tangent at A meets the initial line at N. The finite region R, shown shaded in the diagram, is bounded by C, the line segment AN and the initial line.

b. Use algebraic integration to show that the area of R is exactly $\frac{1}{4}$



(10 marks)

a)

Perpendicular to the initial line: $\frac{dx}{d\theta} = 0$

$$x = e^{\theta}\cos\theta$$

$$\frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta$$

$$e^{\theta}(\cos\theta - \sin\theta) = 0$$

$e^{\theta} > 0$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4}$$

$A(e^{\pi/4}, \frac{\pi}{4})$ (shown)

b)

R = triangle ONA - (polar area from $\theta = 0$ to $\frac{\pi}{4}$)

$$ON = e^{\pi/4}\cos\frac{\pi}{4}$$

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$$AN = e^{\pi/4}\sin\frac{\pi}{4}$$

$$\text{triangle ONA} = \frac{1}{2} \times e^{\pi/2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{4}e^{\pi/2}$$

$$\text{polar area} = \frac{1}{2}\int e^{2\theta}\, d\theta$$

$$\text{polar area} = \left[\frac{1}{4}e^{2\theta}\right]$$

Limits $\theta = 0$ to $\theta = \frac{\pi}{4}$

$$\text{polar area} = \frac{1}{4}e^{\pi/2} - \frac{1}{4}$$

$$R = \frac{1}{4}e^{\pi/2} - \left(\frac{1}{4}e^{\pi/2} - \frac{1}{4}\right)$$

$$\boxed{R = \frac{1}{4} \text{ (shown)}}$$

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11 The curve C has polar equation $r = p + \cos 2\theta$, $0 \leq \theta < 2\pi$, where p is a constant and $p > 1$

a. Show that, at the points on C where the tangent is parallel to the initial line, either $\cos\theta = 0$ or $6\sin^2\theta = p + 1$

b. Hence show that there are exactly six points on C where the tangent is parallel to the initial line if and only if $1 < p < 5$

For the rest of this question, $p = 2$

c. Find the polar coordinates of the six points.

d. Find the exact area enclosed by C.

(11 marks)

a)

Parallel to the initial line: $\frac{dy}{d\theta} = 0$ where $y = r\sin\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$y = (p + 1 - 2\sin^2\theta)\sin\theta$$

$$y = (p + 1)\sin\theta - 2\sin^3\theta$$

$$\frac{dy}{d\theta} = (p + 1)\cos\theta - 6\sin^2\theta\cos\theta$$

$$\frac{dy}{d\theta} = \cos\theta(p + 1 - 6\sin^2\theta) = 0$$

$\cos\theta = 0$ or $6\sin^2\theta = p + 1$ (shown)

b)

$\cos\theta = 0$: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, two points for any p

$$\sin^2\theta = \frac{(p + 1)}{6}$$

Four more points need $0 < \frac{(p + 1)}{6} < 1$

$p + 1 < 6$

$p < 5$

If $p = 5$, $\sin\theta = \pm 1$, the same two points; if $p > 5$ no solutions

Exactly six points if and only if $1 < p < 5$ (shown)

c)

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$: $r = 2 + \cos\pi = 1$

$$\sin^2\theta = \frac{3}{6} = \frac{1}{2}$$

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$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$r = 2 + \cos\frac{\pi}{2} = 2$$

$$\boxed{(1, \frac{\pi}{2}), (1, \frac{3\pi}{2}), (2, \frac{\pi}{4}), (2, \frac{3\pi}{4}), (2, \frac{5\pi}{4}), (2, \frac{7\pi}{4})}$$

d)

$$A = \frac{1}{2}\int (2 + \cos 2\theta)^2 \, d\theta$$

$$(2 + \cos 2\theta)^2 = 4 + 4\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta$$

$$A = \frac{1}{2}\left[\frac{9}{2}\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]$$

Limits $\theta = 0$ to $\theta = 2\pi$

$$A = \frac{1}{2} \times 9\pi$$

$$\boxed{A = \frac{9\pi}{2}}$$

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12 The edge of an ornamental pond is modelled by the curve with polar equation

$$r = a + 0.4\cos 4\theta, \quad 0 \le \theta < 2\pi$$

where a is a positive constant and r is measured in metres, as shown in the diagram. The greatest width of the pond, measured along the initial line through the pole, is 5.6 m.

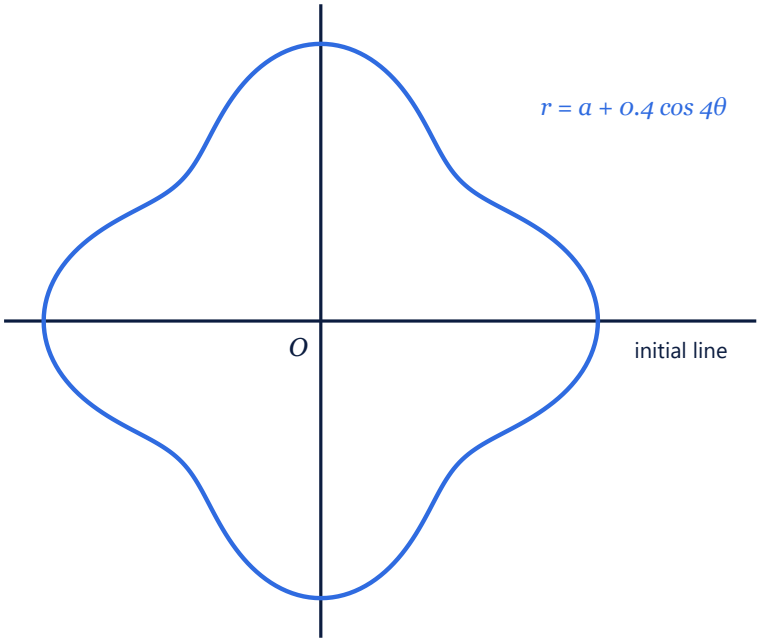
a. Show that $a = 2.4$

b. Use algebraic integration to show that the surface area of the pond, according to the model, is 5.84π m^2

The pond has vertical sides and a horizontal base, and is 0.5 m deep. It is filled using a hosepipe that delivers 25 litres of water per minute. (1 m^3 = 1000 litres.)

c. Find, to the nearest 10 minutes, how long it takes to fill the empty pond.

d. State one reason why the time taken in practice may be different from your answer to part (c).



(10 marks)

a)

$\theta = 0: r = a + 0.4\cos 0 = a + 0.4$

$\theta = \pi: r = a + 0.4\cos 4\pi = a + 0.4$

$2(a + 0.4) = 5.6$

$a + 0.4 = 2.8$

$\boxed{a = 2.4 \text{ (shown)}}$

b)

$A = \frac{1}{2}\int (2.4 + 0.4\cos 4\theta)^2 \, d\theta$

$r^2 = 5.76 + 1.92\cos 4\theta + 0.16\cos^2 4\theta$

$\cos^2 4\theta = \frac{1}{2}(1 + \cos 8\theta)$

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$r^2 = 5.84 + 1.92\cos 4\theta + 0.08\cos 8\theta$

$A = \frac{1}{2}[5.84\theta + 0.48\sin 4\theta + 0.01\sin 8\theta]$

Limits $\theta = 0$ to $\theta = 2\pi$

$A = \frac{1}{2}(5.84 \times 2\pi)$

$A = 5.84\pi \text{ m}^2$ (shown)

c)

$V = 5.84\pi \times 0.5 = 9.1734\ldots \text{ m}^3$

$V = 9173.4\ldots$ litres

time $= 9173.4 \div 25 = 366.9\ldots$ minutes

370 minutes (to the nearest 10 minutes)

d)

The hose flow rate may vary, or the pond is not filled right to the top

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