Calculus of hyperbolic functions

Worked solutions

Further Maths topic practice

This topic online

1 Differentiate each of the following with respect to x.

a. $y = x^2 \cosh 3x$

b. $y = \tanh(x^2)$

c. $y = \ln(\cosh 2x)$

(5 marks)

a) Product rule; $\frac{d}{dx}(\cosh 3x) = 3 \sinh 3x$

$$\frac{dy}{dx} = 2x \cosh 3x + x^2 \times 3 \sinh 3x$$

$$\boxed{\frac{dy}{dx} = 2x \cosh 3x + 3x^2 \sinh 3x}$$

b) Chain rule; $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u$

$$\frac{dy}{dx} = \operatorname{sech}^2(x^2) \times 2x$$

$$\boxed{\frac{dy}{dx} = 2x \operatorname{sech}^2(x^2)}$$

c) Chain rule

$$\frac{dy}{dx} = \frac{1}{\cosh 2x} \times 2 \sinh 2x$$

$$\boxed{\frac{dy}{dx} = 2 \tanh 2x}$$

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2 a. Find $\int (3\cosh 2x - 4\sinh x)\,dx$

b. Find $\int \tanh 2x\,dx$

c. Using the identity $\cosh 2x \equiv 1 + 2\sinh^2 x$, show that

$\int_0^{\ln 3} \sinh^2 x\,dx = \frac{10}{9} - \frac{1}{2}\ln 3$

(8 marks)

a)

$$\int (3\cosh 2x - 4\sinh x)\,dx = \frac{3}{2}\sinh 2x - 4\cosh x + c$$

b)

$$\tanh 2x = \frac{\sinh 2x}{\cosh 2x}$$

The top is $\frac{1}{2}$ x the derivative of the bottom

$$\int \tanh 2x\,dx = \frac{1}{2}\ln(\cosh 2x) + c$$

c) Rearrange the identity

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\int \sinh^2 x\,dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x$$

Limits 0 to ln 3

$$\sinh(2\ln 3) = \sinh(\ln 9) = \frac{(9 - \frac{1}{9})}{2} = \frac{40}{9}$$

$$I = \left(\frac{1}{4} \times \frac{40}{9} - \frac{1}{2}\ln 3\right) - (0 - 0)$$

$$I = \frac{10}{9} - \frac{1}{2}\ln 3$$

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3

a. Given that $y = \text{arsinh}\, x$, prove that $\frac{dy}{dx} = \frac{1}{\sqrt{(1 + x^2)}}$

b. Hence differentiate $\text{arsinh}(3x)$ with respect to x.

c. Given that $\frac{d}{dx}(\text{arcosh}\, x) = \frac{1}{\sqrt{(x^2 - 1)}}$, find $\frac{d}{dx}(\text{arcosh}(x^2))$ for $x > 1$

(7 marks)

a)

$\sinh y = x$

Differentiate with respect to x

$\cosh y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cosh y}$

$\cosh^2 y = 1 + \sinh^2 y$, and $\cosh y > 0$

$\cosh y = \sqrt{(1 + x^2)}$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{(1 + x^2)}}}$$

b) Chain rule with $u = 3x$

$\frac{d}{dx} \text{arsinh}(3x) = \frac{1}{\sqrt{(1 + (3x)^2)}} \times 3$

$$\boxed{\frac{d}{dx} \text{arsinh}(3x) = \frac{3}{\sqrt{(1 + 9x^2)}}}$$

c) Chain rule with $u = x^2$

$\frac{d}{dx} \text{arcosh}(x^2) = \frac{1}{\sqrt{((x^2)^2 - 1)}} \times 2x$

$$\boxed{\frac{d}{dx} \text{arcosh}(x^2) = \frac{2x}{\sqrt{(x^4 - 1)}}}$$

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4 $f(x) = \dfrac{1}{\sqrt{(9x^2 + 16)}}$

a. Show that $\int f(x)\,dx = A\,\text{arsinh}(Bx) + c$, where A and B are constants to be found and c is an arbitrary constant.

b. Hence find the exact value of $\int_0^1 f(x)\,dx$, giving your answer as a multiple of ln 2

c. Find the exact mean value of $f(x)$ over the interval $[0, \frac{4}{3}]$, giving your answer in the form $p\ln(1 + \sqrt{q})$.

(7 marks)

a) Take out the coefficient of x^2

$$\sqrt{(9x^2 + 16)} = 3\sqrt{(x^2 + (\tfrac{4}{3})^2)}$$

$$\int f(x)\,dx = \frac{1}{3}\int \frac{1}{\sqrt{(x^2 + (\tfrac{4}{3})^2)}}\,dx$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}}\,dx = \text{arsinh}(\tfrac{x}{a}) + c$$

$$\int f(x)\,dx = \frac{1}{3}\,\text{arsinh}(\tfrac{3x}{4}) + c$$

$$\boxed{A = \tfrac{1}{3},\ B = \tfrac{3}{4}}$$

b) Limits 0 to 1

$$I = \frac{1}{3}\,\text{arsinh}(\tfrac{3}{4}) - \frac{1}{3}\,\text{arsinh}\,0$$

$$\text{arsinh}(\tfrac{3}{4}) = \ln(\tfrac{3}{4} + \sqrt{(\tfrac{9}{16} + 1)})$$

$$\text{arsinh}(\tfrac{3}{4}) = \ln(\tfrac{3}{4} + \tfrac{5}{4}) = \ln 2$$

$$\boxed{I = \frac{1}{3}\ln 2}$$

c) Mean value $= \frac{1}{(b - a)}\int f(x)\,dx$

Limits 0 to $\frac{4}{3}$

$$\int f(x)\,dx = \frac{1}{3}\,\text{arsinh}\,1 = \frac{1}{3}\ln(1 + \sqrt{2})$$

$$\text{mean} = \frac{3}{4} \times \frac{1}{3}\ln(1 + \sqrt{2})$$

$$\boxed{\text{mean} = \frac{1}{4}\ln(1 + \sqrt{2})}$$

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5 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

Find the exact value of

$$\int_{18}^{23} \frac{1}{\sqrt{(x^2 - 6x - 135)}}\,dx$$

giving your answer in the form ln k, where k is a rational number. (6 marks)

Complete the square

$$x^2 - 6x - 135 = (x - 3)^2 - 9 - 135$$

$$x^2 - 6x - 135 = (x - 3)^2 - 12^2$$

$$\int \frac{1}{\sqrt{(u^2 - a^2)}}\,du = \operatorname{arcosh}\left(\frac{u}{a}\right) + c$$

$$\int \frac{1}{\sqrt{((x - 3)^2 - 12^2)}}\,dx = \operatorname{arcosh}\left(\frac{(x - 3)}{12}\right) + c$$

Limits 18 to 23

$$I = \operatorname{arcosh}\left(\frac{20}{12}\right) - \operatorname{arcosh}\left(\frac{15}{12}\right)$$

$$I = \operatorname{arcosh}\left(\frac{5}{3}\right) - \operatorname{arcosh}\left(\frac{5}{4}\right)$$

$\operatorname{arcosh} x = \ln(x + \sqrt{(x^2 - 1)})$

$$\operatorname{arcosh}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \frac{4}{3}\right) = \ln 3$$

$$\operatorname{arcosh}\left(\frac{5}{4}\right) = \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \ln 2$$

$$I = \ln 3 - \ln 2$$

$$\boxed{I = \ln\left(\frac{3}{2}\right)}$$

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6 The curve C has equation

$y = 6\cosh x - \sinh 2x, \quad x \in \mathbb{R}$

a. Show that the x coordinates of the stationary points of C satisfy

$2\sinh^2 x - 3\sinh x + 1 = 0$

b. Hence find the exact coordinates of the stationary points of C, giving the x coordinates as natural logarithms.

c. Determine the nature of each stationary point. (9 marks)

a)

$$\frac{dy}{dx} = 6\sinh x - 2\cosh 2x$$

$\cosh 2x = 1 + 2\sinh^2 x$

$$\frac{dy}{dx} = 6\sinh x - 2 - 4\sinh^2 x$$

Stationary points: $\frac{dy}{dx} = 0$

$$4\sinh^2 x - 6\sinh x + 2 = 0$$

$$\boxed{2\sinh^2 x - 3\sinh x + 1 = 0}$$

b)

$$(2\sinh x - 1)(\sinh x - 1) = 0$$

$$\sinh x = \frac{1}{2} \text{ or } \sinh x = 1$$

$\operatorname{arsinh} k = \ln(k + \sqrt{(k^2 + 1)})$

$$x = \ln\left(\frac{1}{2} + \sqrt{\left(\frac{5}{4}\right)}\right) = \ln\left(\frac{(1 + \sqrt{5})}{2}\right)$$

$$x = \ln(1 + \sqrt{2})$$

$y = 6\cosh x - 2\sinh x \cosh x = \cosh x(6 - 2\sinh x)$

$$\sinh x = \frac{1}{2} : \cosh x = \frac{\sqrt{5}}{2}, \ y = \frac{\sqrt{5}}{2} \times 5 = \frac{5\sqrt{5}}{2}$$

$$\sinh x = 1 : \cosh x = \sqrt{2}, \ y = \sqrt{2} \times 4 = 4\sqrt{2}$$

$$\boxed{\left(\ln\left(\frac{(1 + \sqrt{5})}{2}\right), \frac{5\sqrt{5}}{2}\right) \text{ and } (\ln(1 + \sqrt{2}), 4\sqrt{2})}$$

c)

$$\frac{d^2y}{dx^2} = 6\cosh x - 4\sinh 2x = \cosh x(6 - 8\sinh x)$$

$$\sinh x = \frac{1}{2} : \frac{d^2y}{dx^2} = \frac{\sqrt{5}}{2} \times 2 = \sqrt{5} > 0$$

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$\sinh x = 1$: $\frac{d^2y}{dx^2} = \sqrt{2} \times (-2) = -2\sqrt{2} < 0$

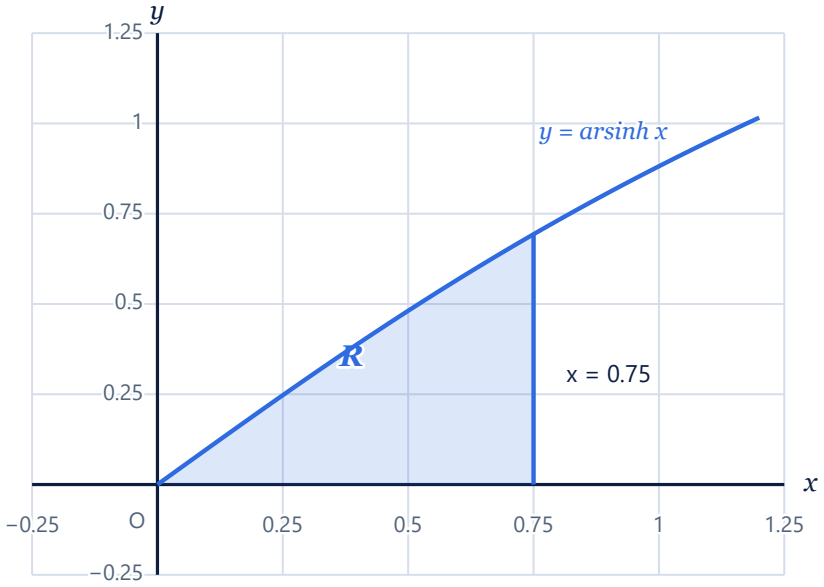
minimum at $\left(\ln\left(\frac{(1+\sqrt{5})}{2}\right), \frac{5\sqrt{5}}{2}\right)$; maximum at $\left(\ln(1+\sqrt{2}), 4\sqrt{2}\right)$

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7 The diagram shows part of the curve with equation $y = \text{arsinh}\, x$, $x \geq 0$

The finite region R, shown shaded, is bounded by the curve, the x-axis and the line with equation $x = \frac{3}{4}$

Use integration by parts to find the exact area of R, giving your answer in the form $a \ln 2 + b$, where a and b are rational numbers.



(6 marks)

Integration by parts: $u = \text{arsinh}\, x$, $\frac{dv}{dx} = 1$

$$\frac{du}{dx} = \frac{1}{\sqrt{(1 + x^2)}}$$

$$v = x$$

$$\int \text{arsinh}\, x \, dx = x \,\text{arsinh}\, x - \int \frac{x}{\sqrt{(1 + x^2)}} \, dx$$

Reverse chain rule

$$\int \text{arsinh}\, x \, dx = x \,\text{arsinh}\, x - \sqrt{(1 + x^2)} + c$$

Limits 0 to $\frac{3}{4}$

$$\text{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$$

$$\sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$$

$$\text{area} = \left(\frac{3}{4} \ln 2 - \frac{5}{4}\right) - (0 - 1)$$

$$\boxed{\text{area} = \frac{3}{4} \ln 2 - \frac{1}{4}}$$

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8 Use the substitution $x = 2 \sinh u$ to show that

$\int_0^2 \sqrt{(x^2 + 4)}\, dx = 2\sqrt{2} + 2 \ln(1 + \sqrt{2})$

(7 marks)

$x = 2 \sinh u$

$\frac{dx}{du} = 2 \cosh u$

$\sqrt{(x^2 + 4)} = \sqrt{(4\sinh^2 u + 4)} = 2 \cosh u$

Change the limits

$x = 0: u = 0$

$x = 2: \sinh u = 1, u = \operatorname{arsinh} 1 = \ln(1 + \sqrt{2})$

$I = \int 2 \cosh u \times 2 \cosh u \, du$

$I = \int 4\cosh^2 u \, du$

$\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$

$I = \int (2 + 2 \cosh 2u) \, du$

$[2u + \sinh 2u] = [2u + 2 \sinh u \cosh u]$

Limits 0 to $\ln(1 + \sqrt{2})$; at the top $\sinh u = 1$, $\cosh u = \sqrt{(1 + 1)} = \sqrt{2}$

$I = (2 \ln(1 + \sqrt{2}) + 2 \times 1 \times \sqrt{2}) - (0 + 0)$

$\boxed{I = 2\sqrt{2} + 2 \ln(1 + \sqrt{2})}$

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9

a. Find $\int \frac{x+6}{\sqrt{(x^2+9)}}\,dx$

b. Hence find the exact value of

$\int_0^4 \frac{x+6}{\sqrt{(x^2+9)}}\,dx$

giving your answer in the form $a + b \ln 3$, where a and b are integers.

(7 marks)

a) Split the fraction

$$\frac{(x+6)}{\sqrt{(x^2+9)}} = \frac{x}{\sqrt{(x^2+9)}} + \frac{6}{\sqrt{(x^2+9)}}$$

Reverse chain rule: $\frac{d}{dx}\sqrt{(x^2+9)} = \frac{x}{\sqrt{(x^2+9)}}$

$$\int \frac{x}{\sqrt{(x^2+9)}}\,dx = \sqrt{(x^2+9)}$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right), \text{ with } a = 3$$

$$\int \frac{6}{\sqrt{(x^2+9)}}\,dx = 6\operatorname{arsinh}\left(\frac{x}{3}\right)$$

$$\boxed{\sqrt{(x^2+9)} + 6\operatorname{arsinh}\left(\frac{x}{3}\right) + c}$$

b) Limits 0 to 4

$$I = \left(\sqrt{25} + 6\operatorname{arsinh}\left(\frac{4}{3}\right)\right) - \left(\sqrt{9} + 6\operatorname{arsinh} 0\right)$$

$$\operatorname{arsinh}\left(\frac{4}{3}\right) = \ln\left(\frac{4}{3} + \sqrt{\left(\frac{16}{9}+1\right)}\right) = \ln\left(\frac{4}{3} + \frac{5}{3}\right) = \ln 3$$

$$I = 5 + 6\ln 3 - 3$$

$$\boxed{I = 2 + 6\ln 3}$$

10 In this question $-\frac{\pi}{2} < x < \frac{\pi}{2}$

a. Show that $\frac{d}{dx}(\text{arsinh}(\tan x)) = \sec x$

b. Hence find the exact value of $\int_0^{\pi/4} \sec x \, dx$, giving your answer as a natural logarithm.

c. Given that $\frac{d}{dx}(\text{artanh } x) = \frac{1}{1 - x^2}$, show that $\frac{d}{dx}(\text{artanh}(\sin x)) = \sec x$

d. Hence prove that $\text{artanh}(\sin x) = \text{arsinh}(\tan x)$

(9 marks)

a) Chain rule

$$\frac{d}{dx}\text{arsinh}(\tan x) = \frac{\sec^2 x}{\sqrt{(1 + \tan^2 x)}}$$

$1 + \tan^2 x = \sec^2 x$, and $\sec x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\frac{d}{dx}\text{arsinh}(\tan x) = \frac{\sec^2 x}{\sec x}$$

$$\boxed{\frac{d}{dx}\text{arsinh}(\tan x) = \sec x}$$

b) Limits 0 to $\frac{\pi}{4}$

$$I = \text{arsinh}(\tan \tfrac{\pi}{4}) - \text{arsinh}(\tan 0)$$

$$I = \text{arsinh } 1 - 0$$

$$\boxed{I = \ln(1 + \sqrt{2})}$$

c) Chain rule

$$\frac{d}{dx}\text{artanh}(\sin x) = \frac{\cos x}{(1 - \sin^2 x)}$$

$$\frac{d}{dx}\text{artanh}(\sin x) = \frac{\cos x}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}\text{artanh}(\sin x) = \sec x}$$

d) Let $g(x) = \text{artanh}(\sin x) - \text{arsinh}(\tan x)$

$$g'(x) = \sec x - \sec x = 0$$

so $g(x)$ is constant on the interval

$$g(0) = \text{artanh } 0 - \text{arsinh } 0 = 0$$

$$\boxed{\text{artanh}(\sin x) = \text{arsinh}(\tan x)}$$

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11 A skydiver jumps from a hovering helicopter. Her speed v m s^{-1}, t seconds after she jumps, is modelled by

$v = 40\tanh(\frac{t}{4}), \quad t \geq 0$

a. State the speed that the skydiver approaches according to the model.

b. Show that her acceleration a m s^{-2} is given by $a = 10 - \frac{v^2}{160}$, and hence find her acceleration when her speed is 30 m s^{-1}

c. Find the exact time at which her speed reaches 30 m s^{-1}, and give this time to 3 significant figures.

d. Find, to 3 significant figures, the distance she falls in this time.

(11 marks)

a) $\tanh(\frac{t}{4}) \to 1$ as $t \to \infty$

$\boxed{40 \text{ m s}^{-1}}$

b)

$$a = \frac{dv}{dt} = 40 \times \frac{1}{4}\operatorname{sech}^2(\frac{t}{4})$$

$$a = 10\operatorname{sech}^2(\frac{t}{4})$$

$\operatorname{sech}^2 = 1 - \tanh^2$

$$a = 10(1 - \tanh^2(\frac{t}{4}))$$

$$\tanh(\frac{t}{4}) = \frac{v}{40}$$

$$a = 10(1 - \frac{v^2}{1600})$$

$$a = 10 - \frac{v^2}{160}$$

$v = 30$

$$a = 10 - \frac{900}{160}$$

$$\boxed{a = \frac{35}{8} = 4.375 \text{ m s}^{-2}}$$

c)

$$40\tanh(\frac{t}{4}) = 30$$

$$\tanh(\frac{t}{4}) = \frac{3}{4}$$

$\operatorname{artanh} x = \frac{1}{2}\ln(\frac{(1+x)}{(1-x)})$

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$$\frac{t}{4} = \frac{1}{2}\ln\left(\frac{(7}{4)/(1/4)}\right) = \frac{1}{2}\ln 7$$

$$t = 2\ln 7$$

$$\boxed{t = 3.89\ \text{s}}$$

d) Distance $= \int v\,dt$

$$\int 40\tanh\left(\frac{t}{4}\right)dt = 160\ln\left(\cosh\left(\frac{t}{4}\right)\right) + c$$

Limits 0 to $2\ln 7$; $\ln(\cosh 0) = 0$

$$\cosh\left(\frac{1}{2}\ln 7\right) = \frac{\left(\sqrt{7} + \frac{1}{\sqrt{7}}\right)}{2} = \frac{4}{\sqrt{7}}$$

$$s = 160\ln\left(\frac{4}{\sqrt{7}}\right) - 0$$

$$\boxed{s = 66.1\ \text{m}}$$

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12 The diagram shows part of the curve H with equation $x^2 - y^2 = 1$, $x \geq 1$, $y \geq 0$, which meets the x-axis at $A(1, 0)$.

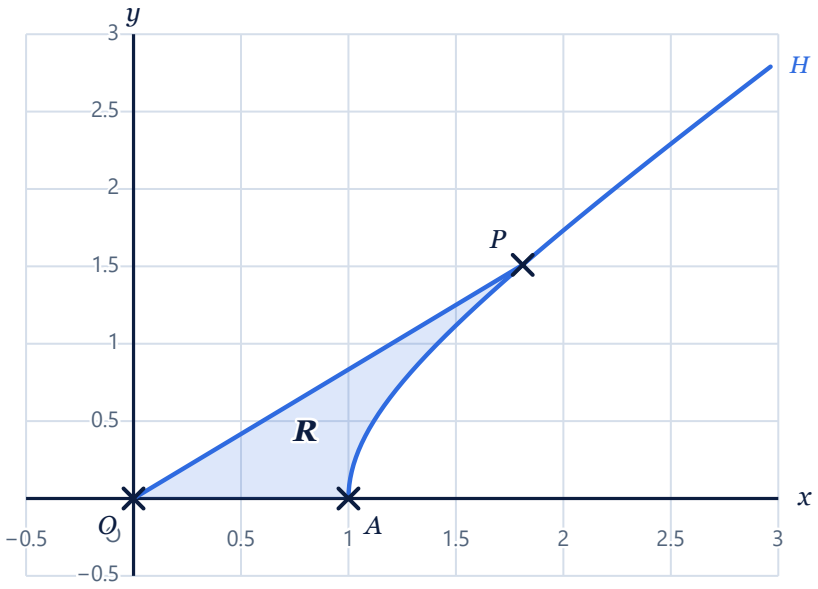
The point $P(\cosh t, \sinh t)$, where $t > 0$, lies on H. The region R, shown shaded, is bounded by the line segment OP, the curve H from A to P, and the line segment OA.

a. Use the substitution $x = \cosh u$ to show that

$$\int \sqrt{(x^2 - 1)}\, dx = \frac{1}{2}x\sqrt{(x^2 - 1)} - \frac{1}{2}\operatorname{arcosh} x + c$$

b. Hence show that the area of R is $\frac{t}{2}$

c. Find the exact coordinates of P when the area of R is ln 2



(11 marks)

a)

$$x = \cosh u$$

$$\frac{dx}{du} = \sinh u$$

$$\sqrt{(x^2 - 1)} = \sqrt{(\cosh^2 u - 1)} = \sinh u$$

$$\int \sqrt{(x^2 - 1)}\, dx = \int \sinh u \times \sinh u\, du$$

$\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$

$$\int \sinh^2 u\, du = \frac{1}{4}\sinh 2u - \frac{1}{2}u + c$$

$$\int \sinh^2 u\, du = \frac{1}{2}\sinh u \cosh u - \frac{1}{2}u + c$$

$\sinh u = \sqrt{(x^2 - 1)}$, $\cosh u = x$, $u = \operatorname{arcosh} x$

$$\boxed{\int \sqrt{(x^2 - 1)}\, dx = \frac{1}{2}x\sqrt{(x^2 - 1)} - \frac{1}{2}\operatorname{arcosh} x + c}$$

b) Let N be $(\cosh t, 0)$: R = triangle ONP − region under H from A to N

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$$\text{triangle } ONP = \frac{1}{2}\cosh t \sinh t$$

Limits 1 to $\cosh t$, with $\int(\cosh^2 t - 1) = \sinh t$ and $\operatorname{arcosh} 1 = 0$

$$\text{area under } H = \left(\frac{1}{2}\cosh t \sinh t - \frac{1}{2}t\right) - (0 - 0)$$

$$R = \frac{1}{2}\cosh t \sinh t - \left(\frac{1}{2}\cosh t \sinh t - \frac{1}{2}t\right)$$

$$\boxed{R = \frac{t}{2}}$$

c)

$$\frac{t}{2} = \ln 2$$

$$t = 2\ln 2 = \ln 4$$

$$\cosh(\ln 4) = \frac{\left(4 + \frac{1}{4}\right)}{2} = \frac{17}{8}$$

$$\sinh(\ln 4) = \frac{\left(4 - \frac{1}{4}\right)}{2} = \frac{15}{8}$$

$$\boxed{P\left(\frac{17}{8}, \frac{15}{8}\right)}$$

Check: $\left(\frac{17}{8}\right)^2 - \left(\frac{15}{8}\right)^2 = \frac{64}{64} = 1$

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