Coupled differential equations

Worked solutions

Further Maths topic practice

This topic online

1 The variables x and y satisfy the differential equations

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = 3x + 4y$$

a. Show that $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 5x = 0$

b. Find a general solution for x.

(5 marks)

a)

Differentiate the first equation

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + \frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + 3x + 4y$$

From the first equation

$$y = \frac{dx}{dt} - 2x$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + 3x + 4\frac{dx}{dt} - 8x$$

$$\boxed{\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 5x = 0}$$

b)

$$m^2 - 6m + 5 = 0$$

$$(m-1)(m-5) = 0$$

$$\boxed{x = Ae^t + Be^{5t}}$$

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2 The variables x and y satisfy the differential equations

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = 3x$$

A general solution for x is $x = Ae^{3t} + Be^{-2t}$, where A and B are constants.

a. Find a general solution for y.

b. Given that $x = 5$ and $y = 0$ when $t = 0$, find the particular solutions for x and y. (5 marks)

a)

Rearrange the first equation

$$y = \frac{1}{2}\left(\frac{dx}{dt} - x\right)$$

$$\frac{dx}{dt} = 3Ae^{3t} - 2Be^{-2t}$$

$$y = \frac{1}{2}(3Ae^{3t} - 2Be^{-2t} - Ae^{3t} - Be^{-2t})$$

$$\boxed{y = Ae^{3t} - \frac{3}{2}Be^{-2t}}$$

b)

$t = 0$: $x = 5$, $y = 0$

$$A + B = 5$$

$$A - \frac{3}{2}B = 0$$

Subtract

$$\frac{5}{2}B = 5$$

$$B = 2, A = 3$$

$$\boxed{x = 3e^{3t} + 2e^{-2t}}$$

$$\boxed{y = 3e^{3t} - 3e^{-2t}}$$

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3 The variables x and y satisfy the differential equations

$$\frac{dx}{dt} = 4x - 2y$$
$$\frac{dy}{dt} = x + y$$

a. Show that $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0$

b. Find general solutions for y and for x. (6 marks)

a)

From the second equation

$$x = \frac{dy}{dt} - y$$

$$\frac{dx}{dt} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substitute into the first equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = 4\left(\frac{dy}{dt} - y\right) - 2y$$

$$\boxed{\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0}$$

b)

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$\boxed{y = Ae^{2t} + Be^{3t}}$$

$$x = \frac{dy}{dt} - y$$

$$x = 2Ae^{2t} + 3Be^{3t} - Ae^{2t} - Be^{3t}$$

$$\boxed{x = Ae^{2t} + 2Be^{3t}}$$

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4 The variables x and y satisfy the differential equations

$$\frac{dx}{dt} = x + ky$$
$$\frac{dy}{dt} = x - y$$

where k is a non-zero constant.

a. Show that $\frac{d^2x}{dt^2} = (k + 1)x$

b. Find the set of values of k for which x oscillates.

Given that $k = -5$, and that $x = 2$ and $y = 0$ when $t = 0$,

c. find x and y in terms of t.

(9 marks)

a)

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + k\frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + kx - ky$$

From the first equation

$$ky = \frac{dx}{dt} - x$$
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + kx - \frac{dx}{dt} + x$$
$$\boxed{\frac{d^2x}{dt^2} = (k+1)x}$$

b)

Oscillation needs imaginary roots of $m^2 = k + 1$

$$k + 1 < 0$$
$$\boxed{k < -1}$$

c)

$$\frac{d^2x}{dt^2} = -4x$$
$$m = \pm 2i$$
$$x = A\cos 2t + B\sin 2t$$

From the first equation with $k = -5$

$$y = \frac{1}{5}\left(x - \frac{dx}{dt}\right)$$
$$y = \frac{1}{5}(A\cos 2t + B\sin 2t + 2A\sin 2t - 2B\cos 2t)$$

$t = 0, x = 2$

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$A = 2$

$t = 0, y = 0$

$$\frac{1}{5}(A - 2B) = 0$$

$$B = 1$$

$$\boxed{x = 2\cos 2t + \sin 2t}$$

$$y = \frac{1}{5}(2\cos 2t + \sin 2t + 4\sin 2t - 2\cos 2t)$$

$$\boxed{y = \sin 2t}$$

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5 The variables x and y satisfy the differential equations

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = -3x - 2y + 3t$$

a. Show that $\frac{d^2x}{dt^2} - x = 3t$

b. Find a general solution for x.

c. Given that $x = 1$ and $y = -2$ when $t = 0$, find the particular solutions for x and y. (10 marks)

a)

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + \frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - 3x - 2y + 3t$$

From the first equation

$$y = \frac{dx}{dt} - 2x$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - 3x - 2\frac{dx}{dt} + 4x + 3t$$

$$\boxed{\frac{d^2x}{dt^2} - x = 3t}$$

b)

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$CF: x = Ae^t + Be^{-t}$$

PI: try $x = \lambda t + \mu$

$$0 - (\lambda t + \mu) = 3t$$

$$\lambda = -3, \mu = 0$$

$$\boxed{x = Ae^t + Be^{-t} - 3t}$$

c)

$$y = \frac{dx}{dt} - 2x$$

$$y = Ae^t - Be^{-t} - 3 - 2Ae^t - 2Be^{-t} + 6t$$

$$y = -Ae^t - 3Be^{-t} + 6t - 3$$

$t = 0$: $x = 1$, $y = -2$

$$A + B = 1$$

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$$-A - 3B - 3 = -2$$
$$A + 3B = -1$$
$$2B = -2$$
$$B = -1, A = 2$$

$$x = 2e^{t} - e^{-t} - 3t$$

$$y = -2e^{t} + 3e^{-t} + 6t - 3$$

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6 The variables x and y satisfy the differential equations

$$\frac{dx}{dt} = 4x - 3y$$
$$\frac{dy}{dt} = 2x - y + 6e^{-t}$$

a. Show that $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = -18e^{-t}$

b. Find a general solution for x.

c. Find a general solution for y.

d. Given that $x = 6$ and $y = 3$ when $t = 0$, find the particular solutions for x and y.

(11 marks)

a)

$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 3\frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 6x + 3y - 18e^{-t}$$

From the first equation

$$3y = 4x - \frac{dx}{dt}$$
$$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 6x + 4x - \frac{dx}{dt} - 18e^{-t}$$

$$\boxed{\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = -18e^{-t}}$$

b)

$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$\text{CF: } x = Ae^t + Be^{2t}$$

PI: try $x = ke^{-t}$

$$ke^{-t} + 3ke^{-t} + 2ke^{-t} = -18e^{-t}$$
$$6k = -18$$
$$k = -3$$

$$\boxed{x = Ae^t + Be^{2t} - 3e^{-t}}$$

c)

$$y = \frac{1}{3}\left(4x - \frac{dx}{dt}\right)$$
$$y = \frac{1}{3}\left(4Ae^t + 4Be^{2t} - 12e^{-t} - Ae^t - 2Be^{2t} - 3e^{-t}\right)$$

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$$y = Ae^{t} + \frac{2}{3}Be^{2t} - 5e^{-t}$$

d)

$t = 0$: $x = 6$, $y = 3$

$$A + B - 3 = 6$$

$$A + \frac{2}{3}B - 5 = 3$$

$$A + B = 9$$

$$A + \frac{2}{3}B = 8$$

Subtract

$$\frac{1}{3}B = 1$$

$$B = 3, A = 6$$

$$x = 6e^{t} + 3e^{2t} - 3e^{-t}$$

$$y = 6e^{t} + 2e^{2t} - 5e^{-t}$$

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7 Two species of algae, P and Q, compete in a pond. At time t days, the population of P is x thousand and the population of Q is y thousand. The populations are modelled by the differential equations

$$\frac{dx}{dt} = 3x - 2y$$
$$\frac{dy}{dt} = 2x - 2y$$

a. Show that $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$

b. Find a general solution for x.

c. Find a general solution for y.

Initially there are 200 thousand of P and 250 thousand of Q. The model predicts that, at time T days, the two populations are equal.

d. Determine the value of T, giving your answer in hours to 3 significant figures.

(11 marks)

a)

$$\frac{d^2x}{dt^2} = 3\frac{dx}{dt} - 2\frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = 3\frac{dx}{dt} - 4x + 4y$$

From the first equation

$$2y = 3x - \frac{dx}{dt}$$
$$4y = 6x - 2\frac{dx}{dt}$$
$$\frac{d^2x}{dt^2} = 3\frac{dx}{dt} - 4x + 6x - 2\frac{dx}{dt}$$

$$\boxed{\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0}$$

b)

$$m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$

$$\boxed{x = Ae^{2t} + Be^{-t}}$$

c)

$$y = \frac{1}{2}\left(3x - \frac{dx}{dt}\right)$$
$$y = \frac{1}{2}(3Ae^{2t} + 3Be^{-t} - 2Ae^{2t} + Be^{-t})$$

$$\boxed{y = \frac{1}{2}Ae^{2t} + 2Be^{-t}}$$

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d)

$t = 0$: $x = 200$, $y = 250$

$$A + B = 200$$

$$\frac{1}{2}A + 2B = 250$$

$$A + 4B = 500$$

$$3B = 300$$

$$B = 100, A = 100$$

$$x = 100e^{2t} + 100e^{-t}$$

$$y = 50e^{2t} + 200e^{-t}$$

Populations equal at $t = T$

$$100e^{2T} + 100e^{-T} = 50e^{2T} + 200e^{-T}$$

$$50e^{2T} = 100e^{-T}$$

$$e^{3T} = 2$$

$$T = \frac{1}{3}\ln 2 \text{ days}$$

$$T = 0.23105 \times 24 \text{ hours}$$

$$\boxed{T = 5.55 \text{ hours}}$$

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8 Two greenhouses are heated by a shared system. At time t hours after a fault is repaired, the temperatures in the two greenhouses are x °C and y °C above their target temperatures. The temperatures are modelled by the differential equations

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -2x - y$$

a. Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$

When $t = 0$, $x = 3$ and $y = 1$

b. Find x and y in terms of t.

c. Show that $x^2 + y^2 = 10e^{-2t}$

d. Hence find a value of T such that, according to the model, both temperatures are within 0.1 °C of their targets for all $t > T$.

(12 marks)

a)

$$\frac{d^2x}{dt^2} = -\frac{dx}{dt} + 2\frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = -\frac{dx}{dt} - 4x - 2y$$

From the first equation

$$2y = \frac{dx}{dt} + x$$
$$\frac{d^2x}{dt^2} = -\frac{dx}{dt} - 4x - \frac{dx}{dt} - x$$
$$\boxed{\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0}$$

b)

$$m^2 + 2m + 5 = 0$$
$$m = -1 \pm 2i$$
$$x = e^{-t}(A\cos 2t + B\sin 2t)$$
$$\frac{dx}{dt} = e^{-t}((2B - A)\cos 2t - (2A + B)\sin 2t)$$
$$y = \frac{1}{2}\left(\frac{dx}{dt} + x\right)$$
$$y = \frac{1}{2}e^{-t}(2B\cos 2t - 2A\sin 2t)$$
$$y = e^{-t}(B\cos 2t - A\sin 2t)$$

$t = 0$: $x = 3$, $y = 1$

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$A = 3, B = 1$

$$\boxed{x = e^{-t}(3\cos 2t + \sin 2t)}$$

$$\boxed{y = e^{-t}(\cos 2t - 3\sin 2t)}$$

c)

$$x^2 + y^2 = e^{-2t}((3\cos 2t + \sin 2t)^2 + (\cos 2t - 3\sin 2t)^2)$$

Expand: the 6 cos 2t sin 2t terms cancel

$$x^2 + y^2 = e^{-2t}(10\cos^2 2t + 10\sin^2 2t)$$

$$\boxed{x^2 + y^2 = 10e^{-2t}}$$

d)

If $x^2 + y^2 < 0.01$ then $|x| < 0.1$ and $|y| < 0.1$

$10e^{-2t} < 0.01$

$e^{2t} > 1000$

$t > \frac{1}{2}\ln 1000$

$$\boxed{t > 3.45 \text{ hours}}$$

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9 A woodland contains red squirrels and grey squirrels. At time t years, there are x hundred red squirrels and y hundred grey squirrels. An ecologist models the populations as follows.

- The rate of change of x is 0.3 times x minus 0.2 times y.
- The rate of change of y is 0.4 times y minus 0.1 times x.

a. Write down a pair of differential equations for this model.

b. Show that $10\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + x = 0$

Initially there are 5000 red squirrels and 4000 grey squirrels.

c. Find x and y in terms of t.

d. Find the time at which, according to the model, the red squirrels die out, and the number of grey squirrels at that time.

e. Comment on the model for the grey squirrels.

(12 marks)

a)

$$\frac{dx}{dt} = 0.3x - 0.2y$$

$$\boxed{\frac{dy}{dt} = 0.4y - 0.1x}$$

b)

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} - 0.2\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} - 0.08y + 0.02x$$

From the first equation

$$0.2y = 0.3x - \frac{dx}{dt}$$

$$0.08y = 0.12x - 0.4\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} - 0.12x + 0.4\frac{dx}{dt} + 0.02x$$

$$\frac{d^2x}{dt^2} = 0.7\frac{dx}{dt} - 0.1x$$

$$\boxed{10\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + x = 0}$$

c)

$$10m^2 - 7m + 1 = 0$$

$$(5m - 1)(2m - 1) = 0$$

$$m = 0.2, m = 0.5$$

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$$x = Ae^{0.5t} + Be^{0.2t}$$

$$y = 1.5x - 5\frac{dx}{dt}$$

$$y = 1.5Ae^{0.5t} + 1.5Be^{0.2t} - 2.5Ae^{0.5t} - Be^{0.2t}$$

$$y = -Ae^{0.5t} + 0.5Be^{0.2t}$$

$t = 0$: $x = 50$, $y = 40$

$$A + B = 50$$

$$-A + 0.5B = 40$$

$$1.5B = 90$$

$$B = 60, A = -10$$

$$\boxed{x = 60e^{0.2t} - 10e^{0.5t}}$$

$$\boxed{y = 10e^{0.5t} + 30e^{0.2t}}$$

d)

Red squirrels die out when $x = 0$

$$60e^{0.2T} = 10e^{0.5T}$$

$$e^{0.3T} = 6$$

$$T = \frac{10}{3}\ln 6$$

$$\boxed{T = 5.97 \text{ years}}$$

$e^{0.5T} = 6^{5/3}$ and $e^{0.2T} = 6^{2/3}$

$$y = 10 \times 6^{5/3} + 30 \times 6^{2/3}$$

$$y = 297.17...$$

$$\boxed{\text{About 29 700 grey squirrels}}$$

e)

$$\boxed{\begin{array}{l}\text{The model predicts the grey squirrels grow exponentially}\\ \text{without limit, which is unrealistic as food and space are}\\ \text{limited}\end{array}}$$

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10 Two lakes, P and Q, are joined by a channel. A river carrying a pollutant flows into lake P. At time t weeks after the pollution starts, the concentration of the pollutant is x mg per litre in lake P and y mg per litre in lake Q. The concentrations are modelled by

$$\frac{dx}{dt} = 2 - 0.4x + 0.1y$$
$$\frac{dy}{dt} = 0.3x - 0.2y$$

Initially there is no pollutant in either lake.

a. Show that $20\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + y = 12$

b. Find a general solution for y.

c. Find the particular solutions for y and for x.

d. Find the concentration in lake Q after 10 weeks, and describe what the model predicts for each lake in the long term.

(12 marks)

a)

From the second equation

$$0.3x = \frac{dy}{dt} + 0.2y$$

$$0.3\frac{dx}{dt} = \frac{d^2y}{dt^2} + 0.2\frac{dy}{dt}$$

Multiply the first equation by 0.3

$$0.3\frac{dx}{dt} = 0.6 - 0.4(0.3x) + 0.03y$$

$$\frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} = 0.6 - 0.4\left(\frac{dy}{dt} + 0.2y\right) + 0.03y$$

$$\frac{d^2y}{dt^2} + 0.6\frac{dy}{dt} + 0.05y = 0.6$$

$$\boxed{20\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + y = 12}$$

b)

$$20m^2 + 12m + 1 = 0$$

$$(10m + 1)(2m + 1) = 0$$

$$m = -0.1,\ m = -0.5$$

PI: try $y = k$

$$k = 12$$

$$\boxed{y = Ae^{-0.1t} + Be^{-0.5t} + 12}$$

c)

$t = 0$: $x = 0$, $y = 0$, so from the second equation $\frac{dy}{dt} = 0$

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$$A + B + 12 = 0$$

$$\frac{dy}{dt} = -0.1Ae^{-0.1t} - 0.5Be^{-0.5t}$$

$$-0.1A - 0.5B = 0$$

$$A = -5B$$

$$-4B = -12$$

$$B = 3,\ A = -15$$

$$\boxed{y = 12 - 15e^{-0.1t} + 3e^{-0.5t}}$$

$$x = \frac{10}{3}\left(\frac{dy}{dt} + 0.2y\right)$$

$$\frac{dy}{dt} + 0.2y = 1.5e^{-0.1t} - 1.5e^{-0.5t} + 2.4 - 3e^{-0.1t} + 0.6e^{-0.5t}$$

$$\frac{dy}{dt} + 0.2y = 2.4 - 1.5e^{-0.1t} - 0.9e^{-0.5t}$$

$$\boxed{x = 8 - 5e^{-0.1t} - 3e^{-0.5t}}$$

d)

$$y = 12 - 15e^{-1} + 3e^{-5}$$

$$\boxed{y = 6.50 \text{ mg per litre}}$$

As $t \to \infty$, the exponential terms $\to 0$

$\boxed{x \to 8 \text{ and } y \to 12 \text{ mg per litre, assuming the river keeps bringing pollutant at the same rate for ever}}$

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11 Two tanks, A and B, are connected by pipes, and salt solution flows into each tank. At time t minutes, tank A contains x kg of salt and tank B contains y kg of salt. The amounts are modelled by

$$\frac{dx}{dt} = -3x + y + 4$$
$$\frac{dy}{dt} = 2x - 2y + 2$$

Initially both tanks contain pure water.

a. Show that $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 10$

b. Find a general solution for x.

c. Find the particular solutions for x and for y.

d. Show that the tanks next contain equal amounts of salt when $u = e^{-t}$ satisfies $u^3 + u^2 + u - 1 = 0$, and hence find this time and the amount of salt in each tank then.

(12 marks)

a)

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + \frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 2x - 2y + 2$$

From the first equation

$$y = \frac{dx}{dt} + 3x - 4$$
$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 2x - 2\frac{dx}{dt} - 6x + 8 + 2$$

$$\boxed{\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 10}$$

b)

$$m^2 + 5m + 4 = 0$$
$$m = -1,\ m = -4$$

PI: try $x = k$

$$4k = 10$$
$$k = 2.5$$

$$\boxed{x = Ae^{-t} + Be^{-4t} + 2.5}$$

c)

$$y = \frac{dx}{dt} + 3x - 4$$
$$y = -Ae^{-t} - 4Be^{-4t} + 3Ae^{-t} + 3Be^{-4t} + 7.5 - 4$$
$$y = 2Ae^{-t} - Be^{-4t} + 3.5$$

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$t = 0$: $x = 0$, $y = 0$

$$A + B = -2.5$$
$$2A - B = -3.5$$
$$3A = -6$$
$$A = -2, B = -0.5$$

$$\boxed{x = 2.5 - 2e^{-t} - 0.5e^{-4t}}$$

$$\boxed{y = 3.5 - 4e^{-t} + 0.5e^{-4t}}$$

d)

$$x - y = -1 + 2e^{-t} - e^{-4t}$$

$x = y$, with $v = e^{-t}$

$$v^4 - 2v + 1 = 0$$
$$(v - 1)(v^3 + v^2 + v - 1) = 0$$

$v = 1$ gives $t = 0$, so $v^3 + v^2 + v - 1 = 0$

$$v = 0.54369$$
$$t = -\ln 0.54369$$

$$\boxed{t = 0.609 \text{ minutes}}$$

$$x = 2.5 - 2(0.54369) - 0.5(0.54369)^4$$

$$\boxed{1.37 \text{ kg of salt in each tank}}$$

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12 On an island, barn owls hunt field voles. At time t years after a study begins, there are x barn owls and y hundred field voles. The populations are modelled by

$$\frac{dx}{dt} = 0.2x + 0.1y$$
$$\frac{dy}{dt} = -0.5x + 0.4y$$

a. Show that $100\frac{d^2x}{dt^2} - 60\frac{dx}{dt} + 13x = 0$

b. Find a general solution for the number of barn owls at time t years.

At the start of the study there are 30 barn owls and 10 000 field voles.

c. Find the particular solutions for x and y.

d. Find the time at which, according to the model, the field voles die out, and the number of barn owls at that time.

e. Comment on the model in the light of your answer to part (d).

(12 marks)

a)

$$\frac{d^2x}{dt^2} = 0.2\frac{dx}{dt} + 0.1\frac{dy}{dt}$$
$$\frac{d^2x}{dt^2} = 0.2\frac{dx}{dt} - 0.05x + 0.04y$$

From the first equation

$$0.1y = \frac{dx}{dt} - 0.2x$$
$$0.04y = 0.4\frac{dx}{dt} - 0.08x$$
$$\frac{d^2x}{dt^2} = 0.6\frac{dx}{dt} - 0.13x$$

$$\boxed{100\frac{d^2x}{dt^2} - 60\frac{dx}{dt} + 13x = 0}$$

b)

$$100m^2 - 60m + 13 = 0$$
$$m = \frac{(60 \pm \sqrt{(3600 - 5200)})}{200}$$
$$m = 0.3 \pm 0.2i$$

$$\boxed{x = e^{0.3t}(A\cos 0.2t + B\sin 0.2t)}$$

c)

$$y = 10\frac{dx}{dt} - 2x$$
$$\frac{dx}{dt} = e^{0.3t}((0.3A + 0.2B)\cos 0.2t + (0.3B - 0.2A)\sin 0.2t)$$

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$$y = e^{0.3t}((3A + 2B - 2A)\cos 0.2t + (3B - 2A - 2B)\sin 0.2t)$$
$$y = e^{0.3t}((A + 2B)\cos 0.2t + (B - 2A)\sin 0.2t)$$

$t = 0$: $x = 30$, $y = 100$

$$A = 30$$
$$A + 2B = 100$$
$$B = 35$$

$$\boxed{x = e^{0.3t}(30\cos 0.2t + 35\sin 0.2t)}$$
$$\boxed{y = e^{0.3t}(100\cos 0.2t - 25\sin 0.2t)}$$

d)

Voles die out when $y = 0$

$$100\cos 0.2T - 25\sin 0.2T = 0$$
$$\tan 0.2T = 4$$
$$T = 5\arctan 4$$

$$\boxed{T = 6.63 \text{ years}}$$

$$\cos 0.2T = \frac{1}{\sqrt{17}}, \sin 0.2T = \frac{4}{\sqrt{17}}$$
$$x = e^{0.3T}\left(\frac{30}{\sqrt{17}} + \frac{140}{\sqrt{17}}\right)$$
$$x = 301.2...$$

$$\boxed{\text{About 301 barn owls}}$$

e)

$$\boxed{\begin{array}{l}\text{With no voles left the owls have no food, yet the model has}\\ \text{their population still increasing: the model is not valid after}\\ \text{this time}\end{array}}$$

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