De Moivre's theorem

Worked solutions

Further Maths topic practice

This topic online

1

a. Use de Moivre's theorem to express $(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9})^6$ in the form $a + bi$, where a and b are exact.

b. Express $\dfrac{(\cos 2\theta + i \sin 2\theta)^3}{(\cos \theta - i \sin \theta)^2}$ in the form $\cos n\theta + i \sin n\theta$, where n is an integer to be found.

(4 marks)

a) $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

$$(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9})^6 = \cos \frac{6\pi}{9} + i \sin \frac{6\pi}{9}$$

$$(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9})^6 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

b) Write the bottom bracket in cos + i sin form

$$\cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta)$$

$$(\cos 2\theta + i \sin 2\theta)^3 = \cos 6\theta + i \sin 6\theta$$

$$(\cos(-\theta) + i \sin(-\theta))^2 = \cos(-2\theta) + i \sin(-2\theta)$$

Dividing: subtract the arguments

$$6\theta - (-2\theta) = 8\theta$$

$\cos 8\theta + i \sin 8\theta$, so $n = 8$

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2 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

Use de Moivre's theorem to find, in the form $a + bi$ where a and b are exact,

a. $(1 - \sqrt{3}\,i)^7$

b. $(-2 + 2i)^{10}$

(4 marks)

a) Modulus-argument form first

$$|1 - \sqrt{3}\,i| = \sqrt{(1 + 3)} = 2$$

$$\arg(1 - \sqrt{3}\,i) = -\frac{\pi}{3}$$

$$(1 - \sqrt{3}\,i)^7 = 2^7\left(\cos\left(-\frac{7\pi}{3}\right) + i\sin\left(-\frac{7\pi}{3}\right)\right)$$

Add 2π to the argument

$$(1 - \sqrt{3}\,i)^7 = 128\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$(1 - \sqrt{3}\,i)^7 = 128\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$\boxed{64 - 64\sqrt{3}\,i}$$

b) Second quadrant

$$|-2 + 2i| = \sqrt{8} = 2\sqrt{2}$$

$$\arg(-2 + 2i) = \frac{3\pi}{4}$$

$$(-2 + 2i)^{10} = (2\sqrt{2})^{10}\left(\cos\frac{30\pi}{4} + i\sin\frac{30\pi}{4}\right)$$

$$(2\sqrt{2})^{10} = 2^{10} \times 2^5 = 32768$$

$$\frac{30\pi}{4} = \frac{15\pi}{2} = 8\pi - \frac{\pi}{2}$$

$$(-2 + 2i)^{10} = 32768\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$\boxed{-32768i}$$

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3 Prove by induction that, for all positive integers n,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (5 marks)

Basis: $n = 1$

$\text{LHS} = (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \text{RHS}$

True for $n = 1$

Assume true for $n = k$

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Consider $n = k + 1$

$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$(\cos\theta + i\sin\theta)^{k+1} = \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

Compound angle formulae

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

So true for $n = k + 1$

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all positive integers n

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4 a. Use de Moivre's theorem to show that
$\cos 4\theta \equiv 8\cos^4\theta - 8\cos^2\theta + 1$

b. Hence find the exact value of $8\cos^4(\frac{\pi}{12}) - 8\cos^2(\frac{\pi}{12})$

(5 marks)

a) Let $c = \cos\theta$, $s = \sin\theta$

$\cos 4\theta + i\sin 4\theta = (c + is)^4$

Binomial expansion

$(c + is)^4 = c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

$(c + is)^4 = c^4 + 4ic^3 s - 6c^2 s^2 - 4ics^3 + s^4$

Equate real parts

$\cos 4\theta = c^4 - 6c^2 s^2 + s^4$

$s^2 = 1 - c^2$

$\cos 4\theta = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2$

$\cos 4\theta = c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4$

$$\boxed{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1}$$

b) The expression is $\cos 4\theta - 1$ with $\theta = \frac{\pi}{12}$

$8\cos^4(\frac{\pi}{12}) - 8\cos^2(\frac{\pi}{12}) = \cos(\frac{\pi}{3}) - 1$

$8\cos^4(\frac{\pi}{12}) - 8\cos^2(\frac{\pi}{12}) = \frac{1}{2} - 1$

$$\boxed{-\frac{1}{2}}$$

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5 $z = \cos\theta + i\sin\theta$

a. Show that, for any positive integer n,

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

b. Hence show that

$$\sin^3\theta \equiv \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

(6 marks)

a) de Moivre with n and with -n

$$z^n = \cos n\theta + i\sin n\theta$$

$$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$

$$\boxed{z^n - \frac{1}{z^n} = 2i\sin n\theta}$$

b) Use n = 1 and cube

$$\left(z - \frac{1}{z}\right)^3 = (2i\sin\theta)^3 = 8i^3\sin^3\theta = -8i\sin^3\theta$$

Expand and pair the terms

$$\left(z - \frac{1}{z}\right)^3 = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$$

$$\left(z - \frac{1}{z}\right)^3 = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$$

$$\left(z - \frac{1}{z}\right)^3 = 2i\sin 3\theta - 6i\sin\theta$$

$$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$$

Divide by -8i

$$\sin^3\theta = \frac{(6\sin\theta - 2\sin 3\theta)}{8}$$

$$\boxed{\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)}$$

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6 a. Use de Moivre's theorem to show that

$\tan 3\theta \equiv \dfrac{3t - t^3}{1 - 3t^2}$

where $t = \tan\theta$.

b. By choosing a suitable value of θ, show that $\tan(\frac{\pi}{12})$ is a root of the equation

$t^3 - 3t^2 - 3t + 1 = 0$

c. Hence find the exact value of $\tan(\frac{\pi}{12})$, giving a reason for your choice of root.

(8 marks)

a) $c = \cos\theta$, $s = \sin\theta$

$\cos 3\theta + i \sin 3\theta = (c + is)^3$

$(c + is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3$

$(c + is)^3 = c^3 + 3ic^2 s - 3cs^2 - is^3$

Equate real and imaginary parts

$\cos 3\theta = c^3 - 3cs^2$

$\sin 3\theta = 3c^2 s - s^3$

$\tan 3\theta = \dfrac{(3c^2 s - s^3)}{(c^3 - 3cs^2)}$

Divide the top and the bottom by c^3

$$\boxed{\tan 3\theta = \frac{(3t - t^3)}{(1 - 3t^2)}}$$

b) $\theta = \frac{\pi}{12}$ gives $3\theta = \frac{\pi}{4}$ and $\tan\frac{\pi}{4} = 1$

$\dfrac{(3t - t^3)}{(1 - 3t^2)} = 1$

$3t - t^3 = 1 - 3t^2$

$$\boxed{t^3 - 3t^2 - 3t + 1 = 0}$$

c) Try $t = -1$: $-1 - 3 + 3 + 1 = 0$, so $(t + 1)$ is a factor

$t^3 - 3t^2 - 3t + 1 = (t + 1)(t^2 - 4t + 1)$

$t = \dfrac{(4 \pm \sqrt{(16 - 4)})}{2} = 2 \pm \sqrt{3}$

$0 < \frac{\pi}{12} < \frac{\pi}{4}$, so $0 < \tan(\frac{\pi}{12}) < 1$

$2 + \sqrt{3} > 1$ and $-1 < 0$, so reject them

$$\boxed{\tan(\frac{\pi}{12}) = 2 - \sqrt{3}}$$

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7 a. Use de Moivre's theorem to show that
$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

b. Hence find the distinct roots of the equation
$32x^5 - 40x^3 + 10x + 1 = 0$
giving your answers to 3 decimal places where appropriate. (10 marks)

a) $c = \cos\theta$, $s = \sin\theta$; $\cos 5\theta$ = real part of $(c + is)^5$

$$(c + is)^5 = c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$$

$$(c + is)^5 = c^5 + 5ic^4 s - 10c^3 s^2 - 10ic^2 s^3 + 5cs^4 + is^5$$

Equate real parts

$$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$$

$s^2 = 1 - c^2$

$$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - 2c^2 + c^4)$$

$$\cos 5\theta = c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$$

$$\boxed{\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$$

b) Let $x = \cos\theta$

$$32x^5 - 40x^3 + 10x = 2\cos 5\theta$$

$$2\cos 5\theta + 1 = 0$$

$$\cos 5\theta = -\frac{1}{2}$$

Take $0 \leq \theta \leq \pi$, so $0 \leq 5\theta \leq 5\pi$

$$5\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$$

$$\theta = \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{8\pi}{15}, \frac{2\pi}{3}, \frac{14\pi}{15}$$

$x = \cos\theta$

$$x = \cos\left(\frac{2\pi}{15}\right) = 0.9135\ldots$$

$$x = \cos\left(\frac{4\pi}{15}\right) = 0.6691\ldots$$

$$x = \cos\left(\frac{8\pi}{15}\right) = -0.1045\ldots$$

$$x = \cos\left(\frac{2\pi}{3}\right) = -0.5$$

$$x = \cos\left(\frac{14\pi}{15}\right) = -0.9781\ldots$$

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$x = 0.914, 0.669, -0.105, -0.5, -0.978$

Five distinct roots: all the roots of the quintic

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8 $z = \cos\theta + i\sin\theta$, and you may use without proof that $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

a. Show that

$\sin^5\theta \equiv \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

b. Hence, making your reasoning clear, solve

$\sin 5\theta - 5\sin 3\theta + 6\sin\theta = 0$

for $0 \le \theta < 2\pi$.

(9 marks)

a) $z - \frac{1}{z} = 2i\sin\theta$, so raise to the power 5

$$\left(z - \frac{1}{z}\right)^5 = (2i)^5\sin^5\theta = 32i\sin^5\theta$$

Binomial expansion, then pair the terms

$$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$\left(z - \frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\left(z - \frac{1}{z}\right)^5 = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

Divide by $32i$

$$\boxed{\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)}$$

b) Write $6\sin\theta$ as $10\sin\theta - 4\sin\theta$

$$\sin 5\theta - 5\sin 3\theta + 10\sin\theta - 4\sin\theta = 0$$

$$16\sin^5\theta - 4\sin\theta = 0$$

$$4\sin\theta(4\sin^4\theta - 1) = 0$$

$$\sin\theta = 0 \text{ or } \sin^4\theta = \frac{1}{4}$$

$\sin^2\theta \ge 0$, so $\sin^2\theta = \frac{1}{2}$ only

$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

$\sin\theta = 0$: $\theta = 0, \pi$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\boxed{\theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}}$$

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9 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

a. Given that $z = \cos\theta + i\sin\theta$, use $z + \frac{1}{z} = 2\cos\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$ to express $\cos^4\theta$ in the form

$a\cos 4\theta + b\cos 2\theta + c$

where a, b and c are rational constants to be found.

b. Hence find the exact value of

$\int_0^{\pi/3} \cos^4\theta \, d\theta$

(7 marks)

a) Raise $z + \frac{1}{z} = 2\cos\theta$ to the power 4

$$\left(z + \frac{1}{z}\right)^4 = 16\cos^4\theta$$

$$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$\left(z + \frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\boxed{\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}}$$

b) Integrate term by term

$$\int \cos^4\theta \, d\theta = \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta$$

Limits 0 to $\frac{\pi}{3}$: the lower limit gives 0

$$I = \frac{1}{32}\sin\frac{4\pi}{3} + \frac{1}{4}\sin\frac{2\pi}{3} + \frac{3}{8} \times \frac{\pi}{3}$$

$$I = \frac{1}{32}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{8}$$

$$I = -\frac{\sqrt{3}}{64} + \frac{8\sqrt{3}}{64} + \frac{\pi}{8}$$

$$\boxed{I = \frac{7\sqrt{3}}{64} + \frac{\pi}{8}}$$

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10 a. Given that $z = \cos\theta + i\sin\theta$, show that

$32\sin^6\theta \equiv 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta$

A glass bead is modelled by the solid formed when the region bounded by the curve with equation $y = \sin^3 x$, $0 \le x \le \pi$, and the x-axis is rotated through 2π radians about the x-axis. The units are centimetres.

b. Use algebraic integration and the result of part (a) to show that the volume of the bead is $\frac{5\pi^2}{16}$ cm^3.

c. The glass has a density of 2.5 grams per cm^3. Find, to 3 significant figures, the total mass of 40 of these beads.

(11 marks)

a) $z - \frac{1}{z} = 2i\sin\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$

$$\left(z - \frac{1}{z}\right)^6 = (2i)^6\sin^6\theta = -64\sin^6\theta$$

Binomial coefficients 1, 6, 15, 20, 15, 6, 1 with alternating signs

$$\left(z - \frac{1}{z}\right)^6 = z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$$

$$\left(z - \frac{1}{z}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

Divide by -2

$$\boxed{32\sin^6\theta = 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta}$$

b) Volume $= \pi\int y^2\,dx$ with $y^2 = \sin^6 x$

$$V = \pi\int \sin^6 x\,dx$$

$$V = \frac{\pi}{32}\int (10 - 15\cos 2x + 6\cos 4x - \cos 6x)\,dx$$

$$V = \frac{\pi}{32}\left[10x - \frac{15}{2}\sin 2x + \frac{3}{2}\sin 4x - \frac{1}{6}\sin 6x\right]$$

Limits 0 to π: every sine term is 0 at both limits

$$V = \frac{\pi}{32}(10\pi - 0)$$

$$\boxed{V = \frac{5\pi^2}{16}\ \text{cm}^3}$$

c) Mass = number × density × volume

$$\text{mass} = 40 \times 2.5 \times \frac{5\pi^2}{16}$$

$$\text{mass} = 308.42\ldots$$

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11 a. Show that, for $\cos\theta \neq 0$,

$$\frac{1 + i\tan\theta}{1 - i\tan\theta} = \cos 2\theta + i\sin 2\theta$$

b. Hence solve the equation

$$\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^5 = -1$$

for $0 < \theta < \frac{\pi}{2}$, giving your answers as exact multiples of π.

(8 marks)

a) Multiply the top and the bottom by $\cos\theta$

$$\frac{(1 + i\tan\theta)}{(1 - i\tan\theta)} = \frac{(\cos\theta + i\sin\theta)}{(\cos\theta - i\sin\theta)}$$

$$\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$$

Dividing: subtract the arguments, $\theta - (-\theta) = 2\theta$

$$\boxed{\frac{(1 + i\tan\theta)}{(1 - i\tan\theta)} = \cos 2\theta + i\sin 2\theta}$$

b) de Moivre

$$(\cos 2\theta + i\sin 2\theta)^5 = \cos 10\theta + i\sin 10\theta$$

$$\cos 10\theta + i\sin 10\theta = -1$$

$$\cos 10\theta = -1 \text{ and } \sin 10\theta = 0$$

$0 < \theta < \frac{\pi}{2}$, so $0 < 10\theta < 5\pi$

$$10\theta = \pi, 3\pi$$

$$\boxed{\theta = \frac{\pi}{10}, \frac{3\pi}{10}}$$

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12 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

a. Use de Moivre's theorem to show that
$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

b. By considering the equation $\cos 6\theta = 0$, explain why the roots of
$32y^3 - 48y^2 + 18y - 1 = 0$
are $\cos^2(\frac{\pi}{12})$, $\cos^2(\frac{\pi}{4})$ and $\cos^2(\frac{5\pi}{12})$.

c. Hence find the exact value of $\cos^2(\frac{\pi}{12})$.

d. Use the product of the roots of the cubic to show that $\cos(\frac{\pi}{12})\cos(\frac{5\pi}{12}) = \frac{1}{4}$

(12 marks)

a) $c = \cos\theta$, $s = \sin\theta$; $\cos 6\theta$ = real part of $(c + is)^6$

Real terms come from even powers of is

$$\cos 6\theta = c^6 + 15c^4(is)^2 + 15c^2(is)^4 + (is)^6$$

$$\cos 6\theta = c^6 - 15c^4 s^2 + 15c^2 s^4 - s^6$$

$s^2 = 1 - c^2$

$$\cos 6\theta = c^6 - 15c^4(1 - c^2) + 15c^2(1 - c^2)^2 - (1 - c^2)^3$$

$$\cos 6\theta = c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6$$

$$\boxed{\cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1}$$

b) Let $y = \cos^2\theta$

$$32y^3 - 48y^2 + 18y - 1 = \cos 6\theta$$

$$\cos 6\theta = 0$$

$$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

$$\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}$$

These give three different values of $\cos^2\theta$, and a cubic has at most three roots

$$\boxed{\text{Roots: } \cos^2(\frac{\pi}{12}), \cos^2(\frac{\pi}{4}), \cos^2(\frac{5\pi}{12})}$$

c) $\cos^2(\frac{\pi}{4}) = \frac{1}{2}$, so $(2y - 1)$ is a factor

$$32y^3 - 48y^2 + 18y - 1 = (2y - 1)(16y^2 - 16y + 1)$$

$$y = \frac{(16 \pm \sqrt{(256 - 64)})}{32} = \frac{(16 \pm 8\sqrt{3})}{32}$$

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$$y = \frac{(2 \pm \sqrt{3})}{4}$$

$\cos^2(\frac{\pi}{12}) > \cos^2(\frac{5\pi}{12})$, so take the larger root

$$\boxed{\cos^2(\frac{\pi}{12}) = \frac{(2+\sqrt{3})}{4}}$$

d) Product of the roots $= -\frac{d}{a} = \frac{1}{32}$

$$\cos^2(\frac{\pi}{12}) \times \frac{1}{2} \times \cos^2(\frac{5\pi}{12}) = \frac{1}{32}$$

$$\cos^2(\frac{\pi}{12}) \cos^2(\frac{5\pi}{12}) = \frac{1}{16}$$

Both angles are acute, so both cosines are positive

$$\boxed{\cos(\frac{\pi}{12}) \cos(\frac{5\pi}{12}) = \frac{1}{4}}$$

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