Exponential form of complex numbers

Worked solutions

Further Maths topic practice

This topic online

1 Express each of these complex numbers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give r and θ as exact values.

a. $z = -2\sqrt{3} + 2i$

b. $w = 5 - 5i$

(3 marks)

a) Find the modulus

$$|z| = \sqrt{((-2\sqrt{3})^2 + 2^2)}$$

$$|z| = \sqrt{16} = 4$$

z is in the second quadrant: find the angle with the negative real axis first

$$\tan\alpha = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$\arg z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\boxed{z = 4e^{5\pi i/6}}$$

b) w is in the fourth quadrant

$$|w| = \sqrt{(5^2 + 5^2)} = \sqrt{50} = 5\sqrt{2}$$

$$\tan\alpha = \frac{5}{5} = 1, \text{ so } \alpha = \frac{\pi}{4}$$

$$\arg w = -\frac{\pi}{4}$$

$$\boxed{w = 5\sqrt{2}\, e^{-\pi i/4}}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

2 Express each of these complex numbers in the form $a + bi$, where a and b are exact real numbers.

a. $6e^{-2\pi i/3}$

b. $\sqrt{2}\, e^{3\pi i/4}$

(3 marks)

Use $re^{i\theta} = r(\cos\theta + i\sin\theta)$

a)

$$6e^{-2\pi i/3} = 6\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$6e^{-2\pi i/3} = 6\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$\boxed{-3 - 3\sqrt{3}i}$$

b)

$$\sqrt{2}\, e^{3\pi i/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$\sqrt{2}\, e^{3\pi i/4} = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$\boxed{-1 + i}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

3 $z = 3e^{\pi i/4}$ and $w = 2e^{\pi i/6}$

In parts (a) and (b), give your answers in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$.

a. Find zw

b. Find $\frac{z}{w}$

c. Show that z^4 is a real number and state its value.

(4 marks)

a) Multiply the moduli, add the arguments

$$zw = 3 \times 2\, e^{i(\pi/4 + \pi/6)}$$

$$\frac{\pi}{4} + \frac{\pi}{6} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{5\pi}{12}$$

$$\boxed{zw = 6e^{5\pi i/12}}$$

b) Divide the moduli, subtract the arguments

$$\frac{z}{w} = \frac{3}{2} e^{i(\pi/4 - \pi/6)}$$

$$\frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{12}$$

$$\boxed{\frac{z}{w} = \frac{3}{2} e^{\pi i/12}}$$

c) $(re^{i\theta})^n = r^n e^{in\theta}$

$$z^4 = 3^4 e^{4\pi i/4}$$

$$z^4 = 81e^{\pi i}$$

$$z^4 = 81(\cos\pi + i\sin\pi) = 81(-1 + 0i)$$

$$\boxed{z^4 = -81, \text{ a real number}}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

4 (a) Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that

(i) $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

(ii) $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

(b) Hence show that

$$\tan\theta = \frac{e^{2i\theta} - 1}{i(e^{2i\theta} + 1)}$$

(4 marks)

a) Replace θ by $-\theta$: cos is even, sin is odd

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

(i) Add

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$\boxed{e^{i\theta} + e^{-i\theta} = 2\cos\theta}$$

(ii) Subtract

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$\boxed{e^{i\theta} - e^{-i\theta} = 2i\sin\theta}$$

b) Make $\cos\theta$ and $\sin\theta$ the subjects

$$\cos\theta = \{(e^{i\theta} + e^{-i\theta})/2\}$$

$$\sin\theta = \{(e^{i\theta} - e^{-i\theta})/2i\}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \{(e^{i\theta} - e^{-i\theta})/i(e^{i\theta} + e^{-i\theta})\}$$

Multiply the top and the bottom by $e^{i\theta}$

$$\boxed{\tan\theta = \frac{(e^{2i\theta} - 1)}{i(e^{2i\theta} + 1)}}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

5 $z_1 = 2e^{\pi i/4}$ and $z_2 = \sqrt{2}\, e^{\pi i/12}$

(a) Write down the exact values of

(i) $|z_1 z_2|$

(ii) arg $(z_1 z_2)$

Given that $w = z_1 z_2$ and that n is a positive integer,

(b) determine the smallest value of n for which w^n is a negative real number, and the value of w^n in this case,

(c) determine the smallest value of n for which arg $(w^n) = 0$, and the value of w^n in this case. (5 marks)

a) Multiply the moduli, add the arguments

$$|z_1 z_2| = 2 \times \sqrt{2}$$

$$\boxed{|z_1 z_2| = 2\sqrt{2}}$$

$$\arg(z_1 z_2) = \frac{\pi}{4} + \frac{\pi}{12} = \frac{3\pi}{12} + \frac{\pi}{12}$$

$$\boxed{\arg(z_1 z_2) = \frac{\pi}{3}}$$

b) $w = 2\sqrt{2}\, e^{\pi i/3}$, so $w^n = (2\sqrt{2})^n e^{n\pi i/3}$

Negative real: the argument must be an odd multiple of π

$$\frac{n\pi}{3} = \pi$$

$$n = 3$$

$$w^3 = (2\sqrt{2})^3 e^{\pi i} = 16\sqrt{2} \times (-1)$$

$$\boxed{n = 3,\ w^3 = -16\sqrt{2}}$$

c) arg 0: the argument must be a multiple of 2π

$$\frac{n\pi}{3} = 2\pi$$

$$n = 6$$

$$w^6 = (2\sqrt{2})^6 e^{2\pi i} = 2^6 \times 8 \times 1$$

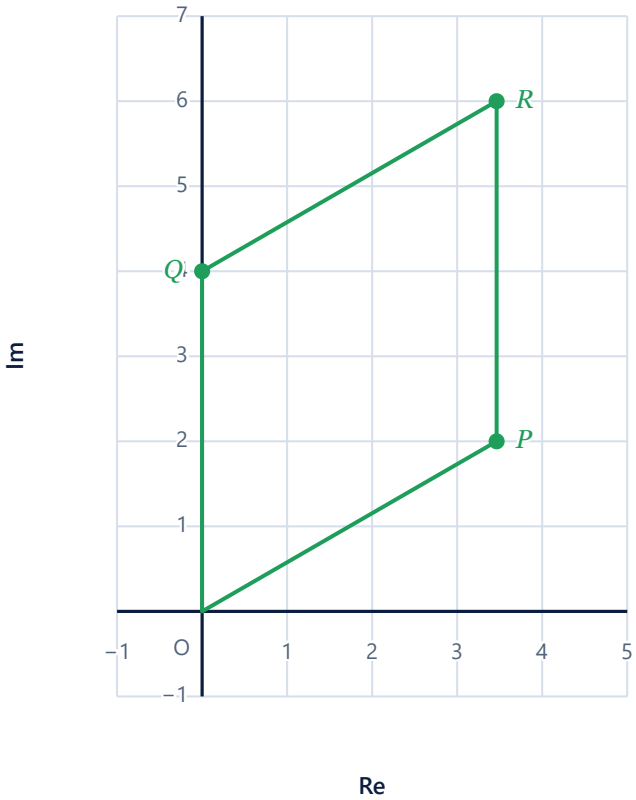
$$\boxed{n = 6,\ w^6 = 512}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

6 $z_1 = 4e^{\pi i/6}$ and $z_2 = 4e^{\pi i/2}$

(a) Show that $z_1 + z_2 = 4\sqrt{3}\, e^{\pi i/3}$

(b) The points P, Q and R represent z_1, z_2 and $z_1 + z_2$ on an Argand diagram, and O is the origin. State the type of quadrilateral $OPRQ$ and explain why arg $(z_1 + z_2)$ lies exactly halfway between arg z_1 and arg z_2.



(5 marks)

a) Write each number as a + bi

$$z1 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{3} + 2i$$

$$z2 = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4i$$

$$z1 + z2 = 2\sqrt{3} + 6i$$

$$|z1 + z2| = \sqrt{((2\sqrt{3})^2 + 6^2)} = \sqrt{48} = 4\sqrt{3}$$

First quadrant

$$\tan\theta = \frac{6}{2\sqrt{3}} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

$$\boxed{z1 + z2 = 4\sqrt{3}\, e^{\pi i/3}}$$

b) $|z1| = |z2| = 4$ and OPRQ is a parallelogram

All four sides are 4, so OPRQ is a rhombus

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

The diagonal OR of a rhombus bisects angle POQ

$\arg(z1 + z2) = \frac{1}{2}(\frac{\pi}{6} + \frac{\pi}{2}) = \frac{\pi}{3}$

Rhombus: OR bisects angle POQ

7 A student was asked to write $\frac{z_1}{z_2}$ in the form $re^{i\theta}$, where $z_1 = -2 + 2i$ and $z_2 = 1 + \sqrt{3}\,i$. The student's attempt is shown below.

Line 1: $|z_1| = 2\sqrt{2}$, $\arg z_1 = \arctan(\frac{2}{-2}) = -\frac{\pi}{4}$

Line 2: $|z_2| = 2$, $\arg z_2 = \frac{\pi}{3}$

Line 3: $\frac{z_1}{z_2} = \frac{2\sqrt{2}}{2}\, e^{i(-\pi/4 + \pi/3)}$

Line 4: $\frac{z_1}{z_2} = \sqrt{2}\, e^{\pi i/12}$

The student made errors in Line 1 and Line 3.

a. Explain and correct the error in (i) Line 1 (ii) Line 3

b. Write down the correct value of $\frac{z_1}{z_2}$ in the form $re^{i\theta}$.

(4 marks)

a)(i) Sketch: -2 + 2i is in the second quadrant

arctan gives $-\pi/4$, which is in the fourth quadrant

$\arg z1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

(ii) For a quotient subtract the arguments

$\frac{z1}{z2} = \frac{2\sqrt{2}}{2} e^{i(3\pi/4 - \pi/3)}$

b)

$\frac{3\pi}{4} - \frac{\pi}{3} = \frac{9\pi}{12} - \frac{4\pi}{12} = \frac{5\pi}{12}$

$\frac{z1}{z2} = \sqrt{2}\, e^{5\pi i/12}$

8 $z = 2e^{\pi i/3}$

a. Write each of z^*, $\frac{1}{z}$ and iz in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$.

b. Show that $z^3 + 8 = 0$

c. Find the smallest positive integer n for which $(iz)^n$ is real, and find the value of $(iz)^n$ in this case. (6 marks)

a) Conjugate: same modulus, opposite argument

$$\boxed{z^* = 2e^{-\pi i/3}}$$

$$\frac{1}{z} = \frac{1e^{0}}{2e^{\pi i/3}} = \frac{1}{2}e^{-\pi i/3}$$

$$\boxed{\frac{1}{z} = \frac{1}{2}e^{-\pi i/3}}$$

$i = e^{\pi i/2}$: multiplying by i rotates by $\frac{\pi}{2}$

$$iz = 2e^{i(\pi/3 + \pi/2)}$$

$$\boxed{iz = 2e^{5\pi i/6}}$$

b)

$$z^3 = 2^3 e^{3\pi i/3} = 8e^{\pi i}$$

$$z^3 = 8(\cos\pi + i\sin\pi) = -8$$

$$\boxed{z^3 + 8 = -8 + 8 = 0}$$

c) Real when the argument is a multiple of π

$$(iz)^n = 2^n e^{5n\pi i/6}$$

Need $\frac{5n}{6}$ to be an integer, so n is a multiple of 6

$$(iz)^6 = 64e^{5\pi i}$$

$$(iz)^6 = 64(\cos 5\pi + i\sin 5\pi) = -64$$

$$\boxed{n = 6, (iz)^6 = -64}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

9

a. By writing each bracket in exponential form, show that

$$\frac{\cos 4\theta + i \sin 4\theta}{\cos 3\theta - i \sin 3\theta} = \cos 7\theta + i \sin 7\theta$$

b. Hence find all the values of θ, in the interval $0 < \theta < \pi$, for which $\frac{\cos 4\theta + i \sin 4\theta}{\cos 3\theta - i \sin 3\theta}$ is real.

(5 marks)

a) cos is even, sin is odd

$$\cos 4\theta + i \sin 4\theta = e^{4i\theta}$$

$$\cos 3\theta - i \sin 3\theta = \cos(-3\theta) + i \sin(-3\theta) = e^{-3i\theta}$$

$$\frac{e^{4i\theta}}{e^{-3i\theta}} = e^{4i\theta + 3i\theta} = e^{7i\theta}$$

$$= \cos 7\theta + i \sin 7\theta$$

b) Real when the imaginary part is zero

$$\sin 7\theta = 0$$

$$7\theta = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$$

$0 < 7\theta < 7\pi$, so 0 and 7π are not included

$$\theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

10 a. Show that $1 + e^{i\theta} = 2\cos(\frac{\theta}{2})\, e^{i\theta/2}$

b. Hence, for $0 < \theta < \pi$, write down the modulus and the argument of $1 + e^{i\theta}$ in terms of θ.

c. Hence show that $(1 + e^{\pi i/3})^6 = -27$

(7 marks)

a) Take out a factor of $e^{i\theta/2}$

$$1 + e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} + e^{i\theta/2})$$

$e^{ix} + e^{-ix} = 2\cos x$ with $x = \frac{\theta}{2}$

$$\boxed{1 + e^{i\theta} = 2\cos(\frac{\theta}{2})\, e^{i\theta/2}}$$

b) $0 < \frac{\theta}{2} < \frac{\pi}{2}$, so $\cos(\frac{\theta}{2}) > 0$: this is already $re^{i\alpha}$ form

$$\boxed{|1 + e^{i\theta}| = 2\cos(\frac{\theta}{2})}$$

$$\boxed{\arg(1 + e^{i\theta}) = \frac{\theta}{2}}$$

c) Put $\theta = \frac{\pi}{3}$

$$1 + e^{\pi i/3} = 2\cos(\frac{\pi}{6})\, e^{\pi i/6}$$

$$1 + e^{\pi i/3} = 2 \times \frac{\sqrt{3}}{2}\, e^{\pi i/6} = \sqrt{3}\, e^{\pi i/6}$$

$$(1 + e^{\pi i/3})^6 = (\sqrt{3})^6\, e^{6\pi i/6}$$

$$(1 + e^{\pi i/3})^6 = 27e^{\pi i} = 27 \times (-1)$$

$$\boxed{(1 + e^{\pi i/3})^6 = -27}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

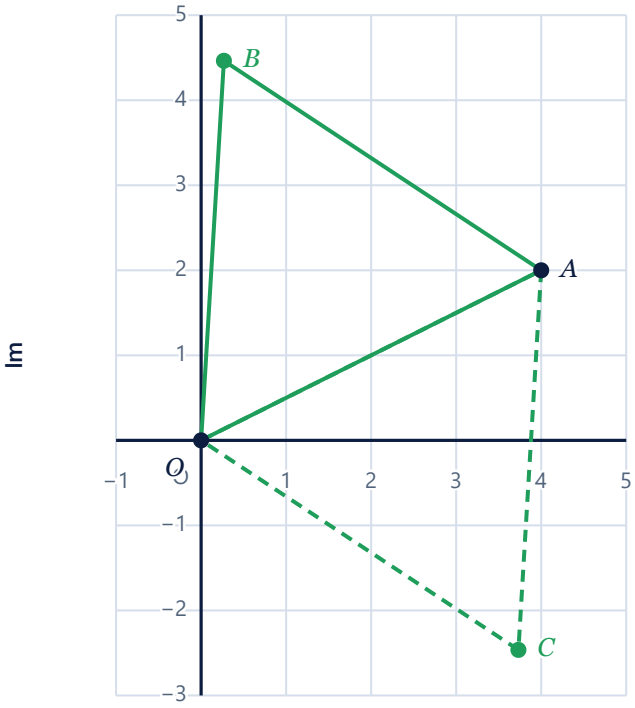
11 The point A on an Argand diagram represents the complex number $4 + 2i$, and O is the origin.

a. Explain why multiplying a complex number by $e^{i\alpha}$ rotates its point about O through an angle α anticlockwise.

b. The triangle OAB is equilateral, and B lies anticlockwise from A about O. Find the complex number represented by B, giving your answer in the form $p + qi$ where p and q are exact.

c. The triangle OAC is also equilateral, with C on the other side of OA from B. Find the complex number represented by C in the same form.

d. Find the exact area of triangle OAB.



(8 marks)

a) Let $z = re^{i\theta}$

$ze^{i\alpha} = re^{i\theta} \times e^{i\alpha} = re^{i(\theta + \alpha)}$

Modulus r is unchanged, the argument increases by α

An anticlockwise rotation through α about O

b) Rotate A through $\frac{\pi}{3}$ about O

$B = (4 + 2i)e^{\pi i/3}$

$B = (4 + 2i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$B = 2 + 2\sqrt{3}\,i + i + \sqrt{3}\,i^2$

$$\boxed{B = (2 - \sqrt{3}) + (1 + 2\sqrt{3})i}$$

c) Rotate A through $-\frac{\pi}{3}$ about O

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$$C = (4 + 2i)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$C = 2 - 2\sqrt{3}\,i + i - \sqrt{3}\,i^2$$

$$\boxed{C = (2 + \sqrt{3}) + (1 - 2\sqrt{3})i}$$

d) Area $= \frac{1}{2}ab\sin C$ with $OA = OB$

$$|OA|^2 = 4^2 + 2^2 = 20$$

$$\text{Area} = \frac{1}{2} \times 20 \times \sin\frac{\pi}{3} = 10 \times \frac{\sqrt{3}}{2}$$

$$\boxed{\text{Area} = 5\sqrt{3}}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

12

a. Show that, for real α and β,

$e^{i\alpha} + e^{i\beta} = 2\cos\left(\frac{\alpha - \beta}{2}\right) e^{i(\alpha + \beta)/2}$

and

$e^{i\alpha} - e^{i\beta} = 2i\sin\left(\frac{\alpha - \beta}{2}\right) e^{i(\alpha + \beta)/2}$

Given that $z_1 = 5e^{5\pi i/12}$ and $z_2 = 5e^{\pi i/12}$,

b. use part (a) to write $z_1 + z_2$ and $z_1 - z_2$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$,

c. hence show that $\frac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary and find its value.

(9 marks)

a) Take out a factor of $e^{i(\alpha + \beta)/2}$

$$e^{i\alpha} + e^{i\beta} = e^{i(\alpha + \beta)/2}\left(e^{i(\alpha - \beta)/2} + e^{-i(\alpha - \beta)/2}\right)$$

$e^{ix} + e^{-ix} = 2\cos x$ and $e^{ix} - e^{-ix} = 2i\sin x$

$$\boxed{e^{i\alpha} + e^{i\beta} = 2\cos\left(\frac{(\alpha - \beta)}{2}\right) e^{i(\alpha + \beta)/2}}$$

$$e^{i\alpha} - e^{i\beta} = e^{i(\alpha + \beta)/2}\left(e^{i(\alpha - \beta)/2} - e^{-i(\alpha - \beta)/2}\right)$$

$$\boxed{e^{i\alpha} - e^{i\beta} = 2i\sin\left(\frac{(\alpha - \beta)}{2}\right) e^{i(\alpha + \beta)/2}}$$

b) $\alpha = \frac{5\pi}{12}$, $\beta = \frac{\pi}{12}$

$$\frac{(\alpha - \beta)}{2} = \frac{4\pi}{24} = \frac{\pi}{6}$$

$$\frac{(\alpha + \beta)}{2} = \frac{6\pi}{24} = \frac{\pi}{4}$$

$$z1 + z2 = 5 \times 2\cos\left(\frac{\pi}{6}\right) e^{\pi i/4} = 10 \times \frac{\sqrt{3}}{2} e^{\pi i/4}$$

$$\boxed{z1 + z2 = 5\sqrt{3}\, e^{\pi i/4}}$$

$$z1 - z2 = 5 \times 2i\sin\left(\frac{\pi}{6}\right) e^{\pi i/4} = 5i\, e^{\pi i/4}$$

$i = e^{\pi i/2}$

$$z1 - z2 = 5e^{\pi i/2} e^{\pi i/4}$$

$$\boxed{z1 - z2 = 5e^{3\pi i/4}}$$

c) Divide the modulo, subtract the arguments

$$\frac{(z1 + z2)}{(z1 - z2)} = \frac{5\sqrt{3}}{5} e^{i(\pi/4 - 3\pi/4)}$$

$$\frac{(z1 + z2)}{(z1 - z2)} = \sqrt{3}\, e^{-\pi i/2}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$$\frac{(z1 + z2)}{(z1 - z2)} = \sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$-\sqrt{3}\,i$, which is purely imaginary

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.