# First order differential equations

Worked solutions

**Further Maths topic practice**

This topic online

**1** Find the general solution of the differential equation

$$\frac{dy}{dx} + 3y = 6e^{2x}$$

giving your answer in the form $y = f(x)$.

(3 marks)

Integrating factor $= e^{\int p\,dx}$

$IF = e^{\int 3\,dx} = e^{3x}$

Multiply through by $e^{3x}$

$$e^{3x}\frac{dy}{dx} + 3ye^{3x} = 6e^{5x}$$

$$\frac{d}{dx}(ye^{3x}) = 6e^{5x}$$

Integrate both sides

$$ye^{3x} = \frac{6}{5}e^{5x} + c$$

$$\boxed{y = \frac{6}{5}e^{2x} + ce^{-3x}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**2** The differential equation

$x\frac{dy}{dx} + 2y = x^3, \quad x > 0$

is to be solved.

a. Find the general solution of the differential equation, giving your answer in the form $y = f(x)$.

b. Find the particular solution for which $y = 1$ when $x = 1$

(5 marks)

a) Divide by $x$ to get the form $\frac{dy}{dx} + Py = Q$

$$\frac{dy}{dx} + \frac{2}{x}y = x^2$$

$$IF = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$$

Multiply through by $x^2$

$$x^2\frac{dy}{dx} + 2xy = x^4$$

$$\frac{d}{dx}(x^2y) = x^4$$

$$x^2y = \frac{x^5}{5} + c$$

$$\boxed{y = \frac{x^3}{5} + \frac{c}{x^2}}$$

b) Substitute $x = 1$, $y = 1$

$$1 = \frac{1}{5} + c$$

$$c = \frac{4}{5}$$

$$\boxed{y = \frac{x^3}{5} + \frac{4}{5x^2}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**3** Find the particular solution of the differential equation

$\frac{dy}{dx} + y \cot x = 2 \cos x, \quad 0 < x < \pi$

for which $y = 3$ when $x = \frac{\pi}{6}$, giving your answer in the form $y = f(x)$.

(5 marks)

$IF = e^{\int \cot x \, dx}$

$IF = e^{\ln \sin x} = \sin x$

Multiply through by $\sin x$

$\sin x \frac{dy}{dx} + y \cos x = 2 \sin x \cos x$

$\frac{d}{dx}(y \sin x) = 2 \sin x \cos x$

Integrate: $\int 2 \sin x \cos x \, dx = \sin^2 x$

$y \sin x = \sin^2 x + c$

$y = \sin x + \frac{c}{\sin x}$

Substitute $x = \frac{\pi}{6}$, $y = 3$

$3 = \frac{1}{2} + \frac{c}{(1/2)}$

$3 = \frac{1}{2} + 2c$

$c = \frac{5}{4}$

$$\boxed{y = \sin x + \frac{5}{4 \sin x}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**4** The curve $C$ has equation $y = f(x)$ and satisfies the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}$$

The curve passes through the point $(0, -2)$.

a. Find $f(x)$.

b. Find the positive value of $x$ for which $y = 0$

c. State what happens to $y$ as $x \to \infty$

(6 marks)

a)

$IF = e^{\int 2x\,dx} = e^{x^2}$

Multiply through by $e^{x^2}$

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = x$$

$$\frac{d}{dx}(ye^{x^2}) = x$$

$$ye^{x^2} = \frac{x^2}{2} + c$$

Substitute $x = 0, y = -2$

$-2 = 0 + c$

$c = -2$

$$\boxed{f(x) = \left(\frac{x^2}{2} - 2\right)e^{-x^2}}$$

b) $e^{-x^2}$ is never 0

$$\frac{x^2}{2} - 2 = 0$$

$$x^2 = 4$$

$$\boxed{x = 2}$$

c) $e^{-x^2} \to 0$ faster than $\frac{x^2}{2} - 2$ grows

$$\boxed{y \to 0 \text{ as } x \to \infty}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**5** Given that

$$\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^3, \quad x > -1$$

and that $y = 5$ when $x = 1$, find $y$ in terms of $x$.

(6 marks)

$$IF = e^{\int -2/(x+1)\,dx}$$

$$IF = e^{-2\ln(x+1)} = \frac{1}{(x+1)^2}$$

Multiply through by $\frac{1}{(x+1)^2}$

$$\frac{1}{(x+1)^2}\frac{dy}{dx} - \frac{2y}{(x+1)^3} = x+1$$

$$\frac{d}{dx}\left(\frac{y}{(x+1)^2}\right) = x+1$$

$$\frac{y}{(x+1)^2} = \frac{(x+1)^2}{2} + c$$

$$y = \frac{(x+1)^4}{2} + c(x+1)^2$$

Substitute $x = 1$, $y = 5$

$$5 = \frac{2^4}{2} + c(2^2)$$

$$5 = 8 + 4c$$

$$c = -\frac{3}{4}$$

$$\boxed{y = \frac{(x+1)^4}{2} - \frac{3}{4}(x+1)^2}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**6** The differential equation

$\frac{dy}{dx} + \frac{y}{x} = \ln x, \quad x > 0$

has a particular solution for which $y = 0$ when $x = 1$

a. Find this particular solution, giving your answer in the form $y = f(x)$.

b. Hence find the exact value of $y$ when $x = e$.

(7 marks)

a)

$$IF = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

Multiply through by $x$

$$x\frac{dy}{dx} + y = x \ln x$$

$$\frac{d}{dx}(xy) = x \ln x$$

$$xy = \int x \ln x \, dx$$

Integrate by parts: $u = \ln x$, $\frac{dv}{dx} = x$

$$\frac{du}{dx} = \frac{1}{x}, \; v = \frac{x^2}{2}$$

$$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx$$

$$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

$$xy = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Substitute $x = 1$, $y = 0$ ($\ln 1 = 0$)

$$0 = 0 - \frac{1}{4} + c$$

$$c = \frac{1}{4}$$

$$\boxed{y = \frac{x \ln x}{2} - \frac{x}{4} + \frac{1}{4x}}$$

b) $\ln e = 1$

$$y = \frac{e}{2} - \frac{e}{4} + \frac{1}{4e}$$

$$\boxed{y = \frac{e}{4} + \frac{1}{4e}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**7** The differential equation

$$\frac{dy}{dx} + \frac{2y}{x(x+2)} = \frac{(x+2)^2}{x}, \quad x > 0$$

is to be solved.

a. Show that an integrating factor for the differential equation is $\frac{x}{x+2}$

b. Hence show that the general solution of the differential equation can be written as

$$y = \frac{(x+2)((x+2)^2 + k)}{2x}$$

where $k$ is an arbitrary constant.

c. Given that $y = 10$ when $x = 2$, find the value of $y$ when $x = 4$

(8 marks)

a) Partial fractions

$$\frac{2}{x(x+2)} = \frac{A}{x} + \frac{B}{(x+2)}$$

$$2 = A(x+2) + Bx$$

$x = 0$: $A = 1$

$x = -2$: $B = -1$

$$\int \frac{2}{x(x+2)}\,dx = \int \left(\frac{1}{x} - \frac{1}{(x+2)}\right) dx$$

$$\int \frac{2}{x(x+2)}\,dx = \ln x - \ln(x+2) = \ln \frac{x}{(x+2)}$$

$$IF = e^{\ln(x/(x+2))}$$

$$\boxed{IF = \frac{x}{(x+2)} \text{ (shown)}}$$

b) Multiply through by the integrating factor

$$\frac{d}{dx}\left(\frac{xy}{(x+2)}\right) = \frac{(x+2)^2}{x} \times \frac{x}{(x+2)}$$

$$\frac{d}{dx}\left(\frac{xy}{(x+2)}\right) = x + 2$$

$$\frac{xy}{(x+2)} = \frac{(x+2)^2}{2} + c$$

Let $c = \frac{k}{2}$

$$\frac{xy}{(x+2)} = \frac{((x+2)^2 + k)}{2}$$

$$\boxed{y = \frac{(x+2)((x+2)^2 + k)}{2x} \text{ (shown)}}$$

c) Substitute $x = 2$, $y = 10$

$$10 = \frac{4(16 + k)}{4}$$

$$16 + k = 10$$

$$k = -6$$

Substitute $x = 4$

$$y = \frac{6(36 - 6)}{8}$$

$$y = \frac{180}{8}$$

$$\boxed{y = \frac{45}{2}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**8** The differential equation

$\frac{dy}{dx} + y \tanh x = 2$

has a particular solution for which $y = 1$ when $x = 0$

a. Show that this particular solution is $y = 2 \tanh x + \operatorname{sech} x$

b. Find the exact value of $y$ when $x = \ln 2$

c. Find the limit of $y$ as $x \to \infty$

(7 marks)

a)

$IF = e^{\int \tanh x \, dx}$

$IF = e^{\ln \cosh x} = \cosh x$

Multiply through by $\cosh x$

$\cosh x \frac{dy}{dx} + y \sinh x = 2 \cosh x$

$\frac{d}{dx}(y \cosh x) = 2 \cosh x$

$y \cosh x = 2 \sinh x + c$

Substitute $x = 0$, $y = 1$

$1 = 0 + c$

$c = 1$

$y = \frac{(2 \sinh x + 1)}{\cosh x}$

$\boxed{y = 2 \tanh x + \operatorname{sech} x \text{ (shown)}}$

b) Use $\cosh x = \frac{(e^x + e^{-x})}{2}$, $\sinh x = \frac{(e^x - e^{-x})}{2}$

$\cosh(\ln 2) = \frac{(2 + 1}{2)/2} = \frac{5}{4}$

$\sinh(\ln 2) = \frac{(2 - 1}{2)/2} = \frac{3}{4}$

$\tanh(\ln 2) = \frac{3}{5}$, $\operatorname{sech}(\ln 2) = \frac{4}{5}$

$y = 2 \times \frac{3}{5} + \frac{4}{5}$

$\boxed{y = 2}$

c) $\tanh x \to 1$ and $\operatorname{sech} x \to 0$

$\boxed{y \to 2}$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**9** A small boat starts from rest. Its velocity, $v$ m s$^{-1}$, $t$ seconds after it starts to move is modelled by the differential equation

$$\frac{dv}{dt} + \frac{2v}{t+1} = k, \quad t \geq 0$$

where $k$ is a positive constant.

a. Solve the differential equation to show that

$$v = \frac{k}{3}\left(t + 1 - \frac{1}{(t+1)^2}\right)$$

Given that the velocity of the boat is 7 m s$^{-1}$ when $t = 1$

b. find the value of $k$,

c. find, according to the model, the velocity of the boat when $t = 3$

d. Comment on the validity of the model for large values of $t$.

(9 marks)

a)

$$IF = e^{\int 2/(t+1)\,dt}$$

$$IF = e^{2\ln(t+1)} = (t+1)^2$$

Multiply through by $(t+1)^2$

$$\frac{d}{dt}\left(v(t+1)^2\right) = k(t+1)^2$$

$$v(t+1)^2 = \frac{k(t+1)^3}{3} + c$$

Starts from rest: $v = 0$ when $t = 0$

$$0 = \frac{k}{3} + c$$

$$c = -\frac{k}{3}$$

$$v(t+1)^2 = \frac{k}{3}(t+1)^3 - \frac{k}{3}$$

$$\boxed{v = \frac{k}{3}\left(t + 1 - \frac{1}{(t+1)^2}\right) \text{ (shown)}}$$

b) Substitute $t = 1$, $v = 7$

$$7 = \frac{k}{3}\left(2 - \frac{1}{4}\right)$$

$$7 = \frac{7k}{12}$$

$$\boxed{k = 12}$$

c) Substitute $t = 3$

$$v = 4\left(4 - \frac{1}{16}\right)$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$v = 16 - \frac{1}{4}$

$v = 15.75 \text{ m s}^{-1}$

d) Large $t$

$v \approx 4(t+1)$, so $v \to \infty$ as $t \to \infty$

Unrealistic: resistance would stop the speed increasing without limit


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**10** A cup of tea is left in a kitchen in which the heating has just been switched off. The temperature of the tea, θ °C, $t$ minutes after it is made, is modelled by assuming that the rate at which the tea cools is proportional to the difference between the temperature of the tea and the temperature of the kitchen.

The temperature of the kitchen, in °C, is modelled as $20 - 0.5t$

The constant of proportionality is 0.2

a. Show that

$$\frac{d\theta}{dt} + 0.2\theta = 4 - 0.1t$$

The tea is made at a temperature of 85 °C.

b. Solve the differential equation to find θ in terms of $t$.

c. Find, according to the model, the temperature of the tea 10 minutes after it is made.

d. Find, according to the model, the time taken for the tea to cool to 40 °C.

e. Give one reason why the model is not suitable for large values of $t$.

(10 marks)

a) Rate of cooling $\propto$ (tea temperature − kitchen temperature)

$$\frac{d\theta}{dt} = -0.2(\theta - (20 - 0.5t))$$

$$\frac{d\theta}{dt} = -0.2\theta + 4 - 0.1t$$

$$\boxed{\frac{d\theta}{dt} + 0.2\theta = 4 - 0.1t \text{ (shown)}}$$

b)

$$IF = e^{\int 0.2\,dt} = e^{0.2t}$$

$$\frac{d}{dt}(\theta e^{0.2t}) = (4 - 0.1t)e^{0.2t}$$

Integrate by parts: $u = 4 - 0.1t$, $\frac{dv}{dt} = e^{0.2t}$

$$\frac{du}{dt} = -0.1,\ v = 5e^{0.2t}$$

$$\int (4 - 0.1t)e^{0.2t}\,dt = 5(4 - 0.1t)e^{0.2t} - \int (-0.1)(5e^{0.2t})\,dt$$

$$\int (4 - 0.1t)e^{0.2t}\,dt = (20 - 0.5t)e^{0.2t} + \int 0.5e^{0.2t}\,dt$$

$$\int (4 - 0.1t)e^{0.2t}\,dt = (20 - 0.5t)e^{0.2t} + 2.5e^{0.2t} + c$$

$$\theta e^{0.2t} = (22.5 - 0.5t)e^{0.2t} + c$$

$$\theta = 22.5 - 0.5t + ce^{-0.2t}$$

Substitute $t = 0$, $\theta = 85$

$$85 = 22.5 + c$$

$$c = 62.5$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$\theta = 22.5 - 0.5t + 62.5e^{-0.2t}$

c) Substitute $t = 10$

$\theta = 22.5 - 5 + 62.5e^{-2}$

$\theta = 17.5 + 8.458...$

$\theta = 26.0\ ^{\circ}C$ (3 s.f.)

d) Solve on a calculator

$22.5 - 0.5t + 62.5e^{-0.2t} = 40$

$t = 5.62$ minutes (3 s.f.)

e)

The kitchen temperature $20 - 0.5t$ keeps falling, reaching $0\ ^{\circ}C$ at $t = 40$, which is unrealistic


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**11** A lake is restocked with fish. The number of fish in the lake, $P$, in thousands, $t$ years after restocking begins, is modelled by the differential equation

$$\frac{dP}{dt} + \frac{2P}{t+5} = 3, \quad t \geq 0$$

Initially there are 2000 fish in the lake.

a. Determine, according to the model, the number of fish in the lake 10 years after restocking begins.

b. Find, according to the model, the rate at which the number of fish is increasing 4 years after restocking begins.

c. State one limitation of the model.

(10 marks)

a)

$$IF = e^{\int 2/(t+5)\,dt} = e^{2\ln(t+5)} = (t+5)^2$$

$$\frac{d}{dt}(P(t+5)^2) = 3(t+5)^2$$

$$P(t+5)^2 = (t+5)^3 + c$$

2000 fish: $P = 2$ when $t = 0$

$$2 \times 25 = 125 + c$$

$$c = -75$$

$$P = t + 5 - \frac{75}{(t+5)^2}$$

Substitute $t = 10$

$$P = 15 - \frac{75}{225}$$

$$P = 14.666...$$

14 667 fish

b) Differentiate $P$

$$\frac{dP}{dt} = 1 + \frac{150}{(t+5)^3}$$

$$\frac{dP}{dt} = 1 + \frac{150}{729} \text{ at } t = 4$$

$$\frac{dP}{dt} = 1.2058 \text{ thousand per year}$$

About 1206 fish per year

c)

$P \to \infty$ as $t \to \infty$, but the lake has limited food and space


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**12** A tank initially holds 200 litres of water in which 10 kg of salt is dissolved.

Salt solution with a concentration of 0.2 kg of salt per litre flows into the tank at a rate of 3 litres per minute. The liquid in the tank is kept well mixed, and liquid flows out of the tank at a rate of 2 litres per minute.

The amount of salt in the tank $t$ minutes after the flows begin is $x$ kg.

a. Show that the amount of salt in the tank is modelled by the differential equation

$$\frac{dx}{dt} = 0.6 - \frac{2x}{200 + t}$$

b. Solve the differential equation to find $x$ in terms of $t$.

c. Find, according to the model, the amount of salt in the tank when the tank holds 300 litres.

d. The tank overflows when it holds 500 litres. Find the concentration of salt in the tank, in kg per litre, at the moment it starts to overflow, and explain why the concentration can never exceed 0.2 kg per litre.

(12 marks)

a) Rate in: 3 litres per minute at 0.2 kg per litre

$$\text{rate in} = 3 \times 0.2 = 0.6 \text{ kg per minute}$$

Volume increases by 3 − 2 = 1 litre per minute

$$V = 200 + t$$

$$\text{concentration} = \frac{x}{(200 + t)}$$

$$\text{rate out} = 2 \times \frac{x}{(200 + t)}$$

$$\boxed{\frac{dx}{dt} = 0.6 - \frac{2x}{(200 + t)} \text{ (shown)}}$$

b) Rearrange

$$\frac{dx}{dt} + \frac{2x}{(200 + t)} = 0.6$$

$$IF = e^{\int 2/(200+t)\,dt} = (200 + t)^2$$

$$\frac{d}{dt}(x(200 + t)^2) = 0.6(200 + t)^2$$

$$x(200 + t)^2 = 0.2(200 + t)^3 + c$$

Substitute $t = 0$, $x = 10$

$$10 \times 200^2 = 0.2 \times 200^3 + c$$

$$400\,000 = 1\,600\,000 + c$$

$$c = -1\,200\,000$$

$$\boxed{x = 0.2(200 + t) - \frac{1\,200\,000}{(200 + t)^2}}$$

c) 300 litres: $200 + t = 300$, so $t = 100$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$$x = 0.2 \times 300 - \frac{1\,200\,000}{300^2}$$

$$x = 60 - 13.33...$$

$$\boxed{x = 46.7 \text{ kg (3 s.f.)}}$$

d) 500 litres: $t = 300$

$$x = 0.2 \times 500 - \frac{1\,200\,000}{500^2}$$

$$x = 100 - 4.8 = 95.2$$

$$\text{concentration} = \frac{95.2}{500}$$

$$\boxed{0.190 \text{ kg per litre (3 s.f.)}}$$

Divide $x$ by the volume

$$\frac{x}{(200 + t)} = 0.2 - \frac{1\,200\,000}{(200 + t)^3}$$

$$\frac{1\,200\,000}{(200 + t)^3} > 0 \text{ for all } t$$

$$\boxed{\begin{array}{l}\text{Concentration} < 0.2 \text{ kg per litre: it approaches the incoming} \\ 0.2 \text{ but never reaches it}\end{array}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.