Hyperbolic functions

Worked solutions

Further Maths topic practice

This topic online

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1 Using the definitions of the hyperbolic functions in terms of exponentials,

a. find the exact value of sinh(ln 3),

b. find the exact value of tanh(ln 2),

c. show that, for $a > 0$, $\cosh(\ln a) = \frac{a^2 + 1}{2a}$

(5 marks)

Use $e^{\ln k} = k$ and $e^{-\ln k} = \frac{1}{k}$

a)

$$\sinh(\ln 3) = \frac{(e^{\ln 3} - e^{-\ln 3})}{2}$$

$$\sinh(\ln 3) = \frac{(3 - \frac{1}{3})}{2}$$

$$\sinh(\ln 3) = \frac{8}{3} \div 2$$

$$\boxed{\sinh(\ln 3) = \frac{4}{3}}$$

b)

$$\tanh x = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

$$\tanh(\ln 2) = \frac{(2 - \frac{1}{2})}{(2 + \frac{1}{2})}$$

$$\tanh(\ln 2) = \frac{3}{2} \div \frac{5}{2}$$

$$\boxed{\tanh(\ln 2) = \frac{3}{5}}$$

c)

$$\cosh(\ln a) = \frac{(e^{\ln a} + e^{-\ln a})}{2}$$

$$\cosh(\ln a) = \frac{(a + \frac{1}{a})}{2}$$

Multiply the top and bottom by a

$$\boxed{\cosh(\ln a) = \frac{(a^2 + 1)}{2a}}$$

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2 Using the definitions of sinh x and cosh x in terms of exponentials,

a. prove that $\cosh^2 x - \sinh^2 x \equiv 1$

b. prove that $\sinh 2x \equiv 2 \sinh x \cosh x$

c. Hence show that $1 - \tanh^2 x \equiv \dfrac{1}{\cosh^2 x}$

(6 marks)

a)

$$\cosh^2 x = \left(\frac{(e^x + e^{-x})}{2}\right)^2$$

$$\cosh^2 x = \frac{(e^{2x} + 2 + e^{-2x})}{4}$$

$$\sinh^2 x = \frac{(e^{2x} - 2 + e^{-2x})}{4}$$

$$\cosh^2 x - \sinh^2 x = \frac{(2 - (-2))}{4}$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

b)

$$2 \sinh x \cosh x = 2 \times \frac{(e^x - e^{-x})}{2} \times \frac{(e^x + e^{-x})}{2}$$

Difference of two squares

$$2 \sinh x \cosh x = \frac{(e^{2x} - e^{-2x})}{2}$$

$$\boxed{2 \sinh x \cosh x = \sinh 2x}$$

c) Divide the identity in (a) by $\cosh^2 x$

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\boxed{1 - \tanh^2 x = \frac{1}{\cosh^2 x}}$$

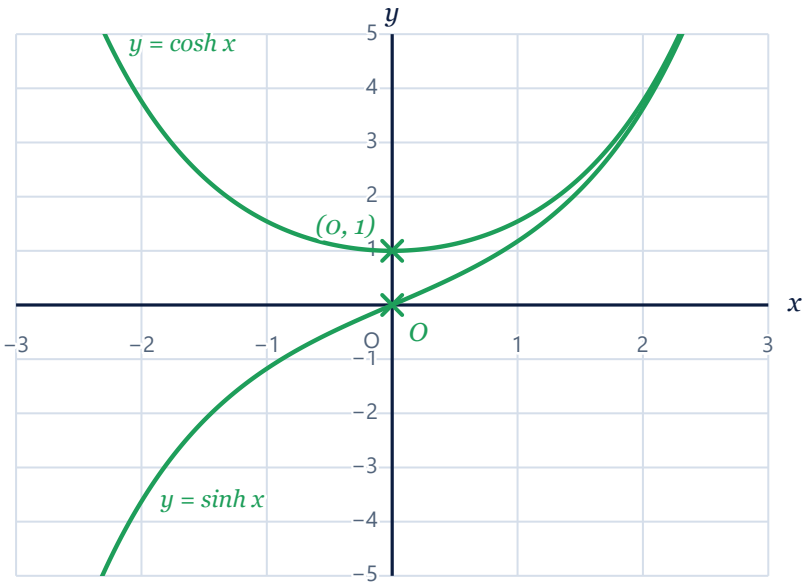
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3

a. On the axes below, sketch the graphs of $y = \cosh x$ and $y = \sinh x$, showing the coordinates of any points where the graphs meet the coordinate axes.

b. Using the definitions in terms of exponentials, show that $\cosh x - \sinh x = e^{-x}$ and $\cosh x + \sinh x = e^{x}$. Hence show that $-1 < \tanh x < 1$ for all real x.

c. Solve the equation $\tanh x = \frac{1}{2}$, giving your answer as an exact natural logarithm.



(8 marks)

a) See the graph: cosh x through (0, 1), sinh x through (0, 0)

cosh x is even with minimum (0, 1); sinh x is odd

Both curves get close to $y = \frac{1}{2}e^{x}$ as x increases

b)

$$\cosh x - \sinh x = \frac{(e^{x} + e^{-x})}{2} - \frac{(e^{x} - e^{-x})}{2}$$

$$\cosh x - \sinh x = e^{-x}$$

$$\cosh x + \sinh x = \frac{(e^{x} + e^{-x})}{2} + \frac{(e^{x} - e^{-x})}{2}$$

$$\cosh x + \sinh x = e^{x}$$

$e^{-x} > 0$ and $e^{x} > 0$ for all x

so $-\cosh x < \sinh x < \cosh x$

Divide by cosh x, which is positive

$$\boxed{-1 < \tanh x < 1}$$

c) Multiply the top and bottom of tanh x by e^{x}

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$$\tanh x = \frac{(e^{2x}-1)}{(e^{2x}+1)}$$

$$\frac{(e^{2x}-1)}{(e^{2x}+1)} = \frac{1}{2}$$

$$2e^{2x} - 2 = e^{2x} + 1$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$\boxed{x = \frac{1}{2}\ln 3}$$

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4 a. Starting from the definition of cosh y in terms of exponentials, show that, for $x \geq 1$,
arcosh $x = \ln(x + \sqrt{(x^2 - 1)})$
explaining why the other root is rejected.

b. Hence find the exact value of $\text{arcosh}(\frac{5}{3})$ in its simplest form.

(6 marks)

a) Let $y = \text{arcosh } x$, so $x = \cosh y$ with $y \geq 0$

$$x = \frac{(e^y + e^{-y})}{2}$$

$$2x = e^y + e^{-y}$$

Multiply by e^y

$$2xe^y = e^{2y} + 1$$

$$e^{2y} - 2xe^y + 1 = 0$$

Quadratic formula in e^y

$$e^y = \frac{(2x \pm \sqrt{(4x^2 - 4)})}{2}$$

$$e^y = x \pm \sqrt{(x^2 - 1)}$$

The roots multiply to 1, so $x - \sqrt{(x^2 - 1)} \leq 1$, giving $y \leq 0$

arcosh $x \geq 0$, so take the + sign

$$y = \ln(x + \sqrt{(x^2 - 1)})$$

$$\boxed{\text{arcosh } x = \ln(x + \sqrt{(x^2 - 1)})}$$

b)

$$\text{arcosh}(\frac{5}{3}) = \ln(\frac{5}{3} + \sqrt{(\frac{25}{9} - 1)})$$

$$\text{arcosh}(\frac{5}{3}) = \ln(\frac{5}{3} + \sqrt{(\frac{16}{9})})$$

$$\text{arcosh}(\frac{5}{3}) = \ln(\frac{5}{3} + \frac{4}{3})$$

$$\boxed{\text{arcosh}(\frac{5}{3}) = \ln 3}$$

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5 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

Solve the equation

$5\cosh x - 3\sinh x = 5$

giving your answers in exact form. (5 marks)

Replace cosh x and sinh x by exponentials

$$5\left(\frac{(e^x + e^{-x})}{2}\right) - 3\left(\frac{(e^x - e^{-x})}{2}\right) = 5$$

Multiply by 2

$$5e^x + 5e^{-x} - 3e^x + 3e^{-x} = 10$$

$$2e^x + 8e^{-x} = 10$$

Multiply by e^x and divide by 2

$$e^{2x} + 4 = 5e^x$$

$$e^{2x} - 5e^x + 4 = 0$$

$$(e^x - 1)(e^x - 4) = 0$$

$$e^x = 1 \text{ or } e^x = 4$$

$$\boxed{x = 0 \text{ or } x = \ln 4}$$

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6 a. Using the definitions of the hyperbolic functions in terms of exponentials, show that
$\cosh 2x \equiv 1 + 2\sinh^2 x$

b. Hence solve the equation
$\cosh 2x - 3\sinh x = 3$
giving your answers as exact natural logarithms.

(8 marks)

a)

$$1 + 2\sinh^2 x = 1 + 2\left(\frac{(e^x - e^{-x})}{2}\right)^2$$

$$1 + 2\sinh^2 x = 1 + \frac{(e^{2x} - 2 + e^{-2x})}{2}$$

$$1 + 2\sinh^2 x = \frac{(e^{2x} + e^{-2x})}{2}$$

$$\boxed{1 + 2\sinh^2 x = \cosh 2x}$$

b) Replace $\cosh 2x$

$$1 + 2\sinh^2 x - 3\sinh x = 3$$

$$2\sinh^2 x - 3\sinh x - 2 = 0$$

$$(2\sinh x + 1)(\sinh x - 2) = 0$$

$$\sinh x = 2 \text{ or } \sinh x = -\frac{1}{2}$$

$\operatorname{arsinh} k = \ln(k + \sqrt{(k^2 + 1)})$

$$x = \ln(2 + \sqrt{(4 + 1)}) = \ln(2 + \sqrt{5})$$

$$x = \ln\left(-\frac{1}{2} + \sqrt{\left(\frac{1}{4} + 1\right)}\right)$$

$$x = \ln\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$

$$\boxed{x = \ln(2 + \sqrt{5}) \text{ or } x = \ln\left(\frac{(\sqrt{5} - 1)}{2}\right)}$$

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7 Let $\alpha = \text{arsinh}(\frac{1}{2})$.

a. Find the exact value of cosh α.

b. Find the exact value of tanh α.

c. Show that sinh 2α = $\frac{\sqrt{5}}{2}$

d. Hence, or otherwise, express 2 arsinh($\frac{1}{2}$) as a single natural logarithm in the form ln($\frac{p + \sqrt{q}}{r}$), where p, q and r are integers.

(7 marks)

$\sinh \alpha = \frac{1}{2}$

a) $\cosh^2 \alpha = 1 + \sinh^2 \alpha$, and $\cosh \alpha > 0$

$\cosh^2 \alpha = 1 + \frac{1}{4} = \frac{5}{4}$

$$\cosh \alpha = \frac{\sqrt{5}}{2}$$

b)

$\tanh \alpha = \frac{\sinh \alpha}{\cosh \alpha}$

$\tanh \alpha = \frac{1}{2} \div \frac{\sqrt{5}}{2}$

$$\tanh \alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

c)

$\sinh 2\alpha = 2 \sinh \alpha \cosh \alpha$

$\sinh 2\alpha = 2 \times \frac{1}{2} \times \frac{\sqrt{5}}{2}$

$$\sinh 2\alpha = \frac{\sqrt{5}}{2}$$

d)

$2\alpha = \text{arsinh}(\frac{\sqrt{5}}{2})$

$2\alpha = \ln(\frac{\sqrt{5}}{2} + \sqrt{(\frac{5}{4} + 1)})$

$2\alpha = \ln(\frac{\sqrt{5}}{2} + \frac{3}{2})$

$$2 \text{arsinh}(\frac{1}{2}) = \ln(\frac{(3 + \sqrt{5})}{2})$$

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Check: $\alpha = \ln\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$, and $\left(\frac{(1+\sqrt{5})}{2}\right)^2 = \frac{(6+2\sqrt{5})}{4} = \frac{(3+\sqrt{5})}{2}$

8 a. Using the definition of cosh x in terms of exponentials, prove that
cosh $3x \equiv 4\cosh^3 x - 3\cosh x$

b. Hence solve the equation
cosh $3x = 13\cosh x$
giving your answers as exact natural logarithms.

(8 marks)

a)

$$4\cosh^3 x = 4\left(\frac{(e^x + e^{-x})}{2}\right)^3$$

$$4\cosh^3 x = \frac{4(e^{3x} + 3e^x + 3e^{-x} + e^{-3x})}{8}$$

$$4\cosh^3 x = \frac{(e^{3x} + 3e^x + 3e^{-x} + e^{-3x})}{2}$$

$$3\cosh x = \frac{(3e^x + 3e^{-x})}{2}$$

$$4\cosh^3 x - 3\cosh x = \frac{(e^{3x} + e^{-3x})}{2}$$

$$\boxed{4\cosh^3 x - 3\cosh x = \cosh 3x}$$

b)

$$4\cosh^3 x - 3\cosh x = 13\cosh x$$

$$4\cosh^3 x - 16\cosh x = 0$$

$$4\cosh x(\cosh^2 x - 4) = 0$$

$\cosh x \geq 1$, so $\cosh x \neq 0$ and $\cosh x \neq -2$

$$\cosh x = 2$$

$\cosh x = k$ has the two solutions $x = \pm \operatorname{arcosh} k$

$$x = \pm \ln(2 + \sqrt{(4-1)})$$

$$\boxed{x = \pm \ln(2 + \sqrt{3})}$$

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9 $f(x) = 3\cosh x - \sinh x, \quad x \in \mathbb{R}$

a. Show that $f(x) = e^x + 2e^{-x}$

b. Solve the equation $f(x) = 3$, giving your answers in exact form.

c. Find the set of values of the constant k for which the equation $f(x) = k$ has real solutions.

d. Hence write down the minimum value of $f(x)$ and find the exact value of x at which it occurs. (10 marks)

a)

$$f(x) = 3\left(\frac{(e^x + e^{-x})}{2}\right) - \frac{(e^x - e^{-x})}{2}$$

$$f(x) = \frac{(3e^x + 3e^{-x} - e^x + e^{-x})}{2}$$

$$f(x) = \frac{(2e^x + 4e^{-x})}{2}$$

$$\boxed{f(x) = e^x + 2e^{-x}}$$

b)

$$e^x + 2e^{-x} = 3$$

Multiply by e^x

$$e^{2x} - 3e^x + 2 = 0$$

$$(e^x - 1)(e^x - 2) = 0$$

$$e^x = 1 \text{ or } e^x = 2$$

$$\boxed{x = 0 \text{ or } x = \ln 2}$$

c)

$$e^x + 2e^{-x} = k$$

$$e^{2x} - ke^x + 2 = 0$$

Real roots for e^x need $b^2 - 4ac \geq 0$

$$k^2 - 8 \geqslant 0$$

Product of roots $= 2 > 0$, so both roots have the same sign

Sum of roots $= k$, which must be positive since $e^x > 0$

$$\boxed{k \geqslant 2\sqrt{2}}$$

d) The least k gives the minimum value, with a repeated root

$$\text{minimum value} = 2\sqrt{2}$$

$$e^x = \frac{k}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$x = \ln\sqrt{2}$$

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minimum $2\sqrt{2}$ at $x = \frac{1}{2}\ln 2$

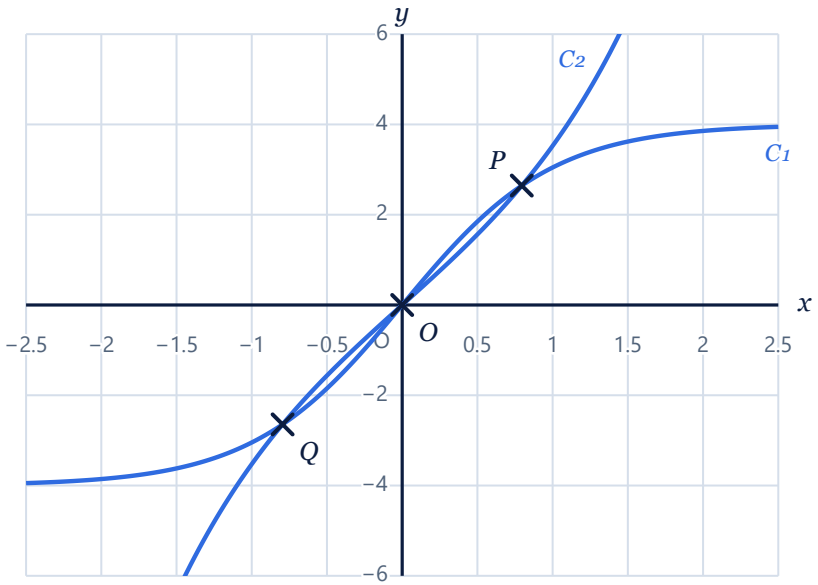
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10 The diagram shows the curves C_1 with equation $y = 4 \tanh x$ and C_2 with equation $y = 3 \sinh x$. The curves meet at the origin O and at the points P and Q.

a. Show that the x coordinates of P and Q satisfy $\cosh x = \frac{4}{3}$

b. Find the exact coordinates of P and of Q, giving the x coordinates as natural logarithms.

c. Hence find the set of values of x for which $3 \sinh x > 4 \tanh x$



(9 marks)

a)

$$3 \sinh x = 4 \tanh x$$

$$3 \sinh x = \frac{4 \sinh x}{\cosh x}$$

$$\sinh x \left(3 - \frac{4}{\cosh x}\right) = 0$$

$\sinh x = 0$ gives $x = 0$, the point O

$$\cosh x = \frac{4}{3} \text{ at P and Q}$$

b)

$$x = \pm \operatorname{arcosh}\left(\frac{4}{3}\right)$$

$$x = \pm \ln\left(\frac{4}{3} + \sqrt{\left(\frac{16}{9} - 1\right)}\right)$$

$$x = \pm \ln\left(\frac{4}{3} + \frac{\sqrt{7}}{3}\right)$$

$$x = \pm \ln\left(\frac{(4 + \sqrt{7})}{3}\right)$$

$\sinh^2 x = \cosh^2 x - 1$, and $\sinh x$ has the same sign as x

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$$\sinh x = \pm\sqrt{\left(\frac{16}{9} - 1\right)} = \pm\frac{\sqrt{7}}{3}$$

$$y = 3\sinh x = \pm\sqrt{7}$$

$$P\left(\ln\left(\frac{(4+\sqrt{7})}{3}\right), \sqrt{7}\right), Q\left(-\ln\left(\frac{(4+\sqrt{7})}{3}\right), -\sqrt{7}\right)$$

c) C_2 is above C_1 to the right of P and between Q and O

$$-\ln\left(\frac{(4+\sqrt{7})}{3}\right) < x < 0 \text{ or } x > \ln\left(\frac{(4+\sqrt{7})}{3}\right)$$

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11 A power cable hangs between the tops of two vertical poles which stand on horizontal ground 60 m apart. The ground is taken as the x-axis and the point on the ground midway between the poles as the origin. The cable is modelled by the curve

$$y = 20\cosh\left(\frac{x}{20}\right) - 15, \quad -30 \le x \le 30$$

where x and y are measured in metres.

a. Find the height of the lowest point of the cable above the ground.

b. Find the height of each pole, giving your answer in terms of e and then to 3 significant figures.

c. Find the exact values of x at which the cable is 10 m above the ground. Hence find, to 3 significant figures, the horizontal distance between these two points.

d. A second engineer models the cable as a parabola $y = ax^2 + b$ with the same lowest point and passing through the tops of both poles. Find, to 3 significant figures, the difference between the heights predicted by the two models at $x = 20$

(10 marks)

a) Lowest point at $x = 0$, where $\cosh 0 = 1$

$$y = 20 - 15$$

$$\boxed{5 \text{ m}}$$

b) $x = 30$

$$y = 20\cosh(1.5) - 15$$

$$y = 20 \times \frac{(e^{1.5} + e^{-1.5})}{2} - 15$$

$$y = 10(e^{1.5} + e^{-1.5}) - 15$$

$$\boxed{y = 32.0 \text{ m}}$$

c)

$$20\cosh\left(\frac{x}{20}\right) - 15 = 10$$

$$\cosh\left(\frac{x}{20}\right) = \frac{25}{20} = \frac{5}{4}$$

$$\frac{x}{20} = \pm \operatorname{arcosh}\left(\frac{5}{4}\right)$$

$$\frac{x}{20} = \pm \ln\left(\frac{5}{4} + \sqrt{\left(\frac{25}{16} - 1\right)}\right)$$

$$\frac{x}{20} = \pm \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \pm \ln 2$$

$$x = \pm 20 \ln 2$$

$$\text{distance} = 40 \ln 2$$

$$\boxed{x = \pm 20 \ln 2, \text{ distance} = 27.7 \text{ m}}$$

d) Same lowest point $(0, 5)$, so $b = 5$

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$$900a + 5 = 32.048\ldots$$

$$a = 0.030054\ldots$$

parabola at $x = 20$: $y = 400a + 5 = 17.021\ldots$

cable at $x = 20$: $y = 20\cosh 1 - 15 = 15.862\ldots$

difference $= 17.021\ldots - 15.862\ldots$

$\boxed{1.16 \text{ m}}$

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12

a. Starting from the definition of $\tanh y$ in terms of exponentials, show that

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$$

b. Show that, for $x > 0$,

$$\operatorname{artanh}\left(\frac{x-1}{x+1}\right) = \frac{1}{2}\ln x$$

c. Hence solve the equation

$$\operatorname{artanh}\left(\frac{x-1}{x+1}\right) = \operatorname{arsinh}\left(\frac{3}{4}\right)$$

(10 marks)

a) Let $y = \operatorname{artanh} x$, so $x = \tanh y$

$$x = \frac{(e^y - e^{-y})}{(e^y + e^{-y})}$$

Multiply the top and bottom by e^y

$$x = \frac{(e^{2y} - 1)}{(e^{2y} + 1)}$$

$$xe^{2y} + x = e^{2y} - 1$$

$$e^{2y}(1 - x) = 1 + x$$

$$e^{2y} = \frac{(1+x)}{(1-x)}$$

$$2y = \ln\left(\frac{(1+x)}{(1-x)}\right)$$

$$\boxed{\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{(1+x)}{(1-x)}\right)}$$

b) For $x > 0$, $|x-1| < x+1$, so $-1 < \frac{(x-1)}{(x+1)} < 1$

$$1 + \frac{(x-1)}{(x+1)} = \frac{2x}{(x+1)}$$

$$1 - \frac{(x-1)}{(x+1)} = \frac{2}{(x+1)}$$

$$\operatorname{artanh}\left(\frac{(x-1)}{(x+1)}\right) = \frac{1}{2}\ln\left(\frac{2x}{(x+1)} \div \frac{2}{(x+1)}\right)$$

$$\boxed{\operatorname{artanh}\left(\frac{(x-1)}{(x+1)}\right) = \frac{1}{2}\ln x}$$

c)

$$\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{9}{16} + 1\right)}\right)$$

$$\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$$

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$$\frac{1}{2}\ln x = \ln 2$$

$$\ln x = 2\ln 2 = \ln 4$$

$$\boxed{x = 4}$$

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