Improper integrals

Worked solutions

Further Maths topic practice

This topic online

1 Evaluate the improper integral

$\int_2^{\infty} 6e^{-3x}\,dx$

giving your answer in exact form. (3 marks)

Replace ∞ by t, then let $t \to \infty$

$\int 6e^{-3x}\,dx = -2e^{-3x} + c$

Evaluate from $x = 2$ to $x = t$

$I(t) = -2e^{-3t} - (-2e^{-6})$

$I(t) = 2e^{-6} - 2e^{-3t}$

As $t \to \infty$, $e^{-3t} \to 0$

$\boxed{I = 2e^{-6}}$

2 $I = \int_0^8 \frac{2}{\sqrt[3]{x}}\,dx$

a. Explain why I is an improper integral.

b. Find the value of I. (4 marks)

a) The integrand is not defined at $x = 0$

$\frac{2}{\sqrt[3]{x}} \to \infty$ as $x \to 0$, and $x = 0$ is the lower limit

b) Write as a power of x

$\frac{2}{\sqrt[3]{x}} = 2x^{-1/3}$

$\int 2x^{-1/3}\,dx = 3x^{2/3} + c$

Integrate from a to 8, then let $a \to 0$

$I(a) = 3(8)^{2/3} - 3a^{2/3}$

$I(a) = 12 - 3a^{2/3}$

As $a \to 0$, $a^{2/3} \to 0$

$\boxed{I = 12}$

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3

a. Evaluate $\int_4^\infty \frac{6}{x\sqrt{x}}\,dx$

b. Show that $\int_4^\infty \frac{6}{\sqrt{x}}\,dx$ is divergent.

(5 marks)

a) Write as a power of x

$$\frac{6}{x\sqrt{x}} = 6x^{-3/2}$$

$$\int 6x^{-3/2}\,dx = -12x^{-1/2} + c$$

Integrate from 4 to t

$$I(t) = -\frac{12}{\sqrt{t}} - \left(-\frac{12}{\sqrt{4}}\right)$$

$$I(t) = 6 - \frac{12}{\sqrt{t}}$$

As $t \to \infty$, $\frac{12}{\sqrt{t}} \to 0$

$$\int = 6$$

b)

$$\int 6x^{-1/2}\,dx = 12x^{1/2} + c$$

Integrate from 4 to t

$$J(t) = 12\sqrt{t} - 12\sqrt{4}$$

$$J(t) = 12\sqrt{t} - 24$$

As $t \to \infty$, $12\sqrt{t} \to \infty$

No finite limit, so the integral is divergent

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4

a. Find the value of $\int_1^5 \frac{3}{\sqrt{(x-1)}}\,dx$

b. Show that $\int_1^5 \frac{1}{(x-1)^2}\,dx$ is divergent.

(6 marks)

Both integrands are undefined at $x = 1$, so integrate from a and let $a \to 1$

a)

$$\int 3(x-1)^{-1/2}\,dx = 6(x-1)^{1/2} + c$$

$$I(a) = 6\sqrt{4} - 6\sqrt{(a-1)}$$

$$I(a) = 12 - 6\sqrt{(a-1)}$$

As $a \to 1$, $\sqrt{(a-1)} \to 0$

$$\boxed{\int = 12}$$

b)

$$\int (x-1)^{-2}\,dx = -\frac{1}{(x-1)} + c$$

$$J(a) = -\frac{1}{4} - \left(-\frac{1}{(a-1)}\right)$$

$$J(a) = \frac{1}{(a-1)} - \frac{1}{4}$$

As $a \to 1$ from above, $a - 1 \to 0$, so $\frac{1}{(a-1)} \to \infty$

No finite limit, so the integral is divergent

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5 Show that the improper integral

$\int_0^{\infty} xe^{-2x}\,dx$

converges, and find its exact value. (6 marks)

Integration by parts: $u = x$, $\frac{dv}{dx} = e^{-2x}$

$$\frac{du}{dx} = 1$$

$$v = -\frac{1}{2}e^{-2x}$$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$$

Integrate from 0 to t

$$I(t) = \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) - \left(0 - \frac{1}{4}\right)$$

$$I(t) = \frac{1}{4} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}$$

As $t \to \infty$, $te^{-2t} \to 0$ because the exponential dominates, and $e^{-2t} \to 0$

The limit exists, so the integral converges

$$\boxed{\int = \frac{1}{4}}$$

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6

a. Explain why $\int_1^\infty \frac{1}{x(2x+3)}\,dx$ is an improper integral.

b. Express $\frac{1}{x(2x+3)}$ in partial fractions.

c. Hence show that $\int_1^\infty \frac{1}{x(2x+3)}\,dx = a \ln b$, where a and b are rational numbers to be found.

(7 marks)

a)

The upper limit is infinite

b) Partial fractions

$$\frac{1}{x(2x+3)} = \frac{A}{x} + \frac{B}{(2x+3)}$$

$$1 = A(2x+3) + Bx$$

$x = 0$: $1 = 3A$, so $A = \frac{1}{3}$

$x = -\frac{3}{2}$: $1 = -\frac{3}{2}B$, so $B = -\frac{2}{3}$

$$\boxed{\frac{1}{x(2x+3)} = \frac{1}{3x} - \frac{2}{3(2x+3)}}$$

c) Integrate term by term

$$\int = \frac{1}{3}\ln x - \frac{1}{3}\ln(2x+3) + c$$

$$\int = \frac{1}{3}\ln\left(\frac{x}{(2x+3)}\right) + c$$

Combine the logs before taking the limit. Integrate from 1 to t

$$I(t) = \frac{1}{3}\ln\left(\frac{t}{(2t+3)}\right) - \frac{1}{3}\ln\left(\frac{1}{5}\right)$$

Divide top and bottom by t

$$\frac{t}{(2t+3)} = \frac{1}{(2+3/t)}$$

$\to \frac{1}{2}$ as $t \to \infty$

$$I = \frac{1}{3}\ln\left(\frac{1}{2}\right) - \frac{1}{3}\ln\left(\frac{1}{5}\right)$$

$$I = \frac{1}{3}\ln\left(\frac{5}{2}\right)$$

$$\boxed{a = \frac{1}{3},\ b = \frac{5}{2}}$$

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7 a. Write $x^2 + 6x + 13$ in the form $(x + p)^2 + q$, where p and q are integers.

b. Hence evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2 + 6x + 13} \, dx$, giving your answer as an exact multiple of π.

(6 marks)

a) Complete the square

$$x^2 + 6x + 13 = (x + 3)^2 + 4$$

b) Standard integral $\int \frac{1}{(u^2 + a^2)} \, du = \frac{1}{a} \arctan\left(\frac{u}{a}\right)$

$$\int \frac{1}{((x + 3)^2 + 2^2)} \, dx = \frac{1}{2} \arctan\left(\frac{(x + 3)}{2}\right) + c$$

Both limits are infinite: split at $x = -3$

From -3 to t

$$\frac{1}{2} \arctan\left(\frac{(t + 3)}{2}\right) - \frac{1}{2} \arctan 0 = \frac{1}{2} \arctan\left(\frac{(t + 3)}{2}\right)$$

As $t \to \infty$, arctan $\to \frac{\pi}{2}$

$$\to \frac{\pi}{4}$$

From s to -3

$$\frac{1}{2} \arctan 0 - \frac{1}{2} \arctan\left(\frac{(s + 3)}{2}\right) = -\frac{1}{2} \arctan\left(\frac{(s + 3)}{2}\right)$$

As $s \to -\infty$, arctan $\to -\frac{\pi}{2}$

$$\to \frac{\pi}{4}$$

Both parts converge, so add them

$$\int = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

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8 $I = \int_0^1 \sqrt{x} \ln x \, dx$

a. Explain why I is an improper integral.

b. Show that $I = -\frac{4}{9}$

You may use without proof the result that $x^k \ln x \to 0$ as $x \to 0$, for any constant $k > 0$ (6 marks)

a)

ln x is not defined at x = 0, the lower limit

b) Integration by parts: $u = \ln x$, $\frac{dv}{dx} = x^{1/2}$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{2}{3}x^{3/2}$$

$$\int x^{1/2} \ln x \, dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2} \times \frac{1}{x} \, dx$$

$$\int x^{1/2} \ln x \, dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2} \, dx$$

$$\int x^{1/2} \ln x \, dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + c$$

Integrate from a to 1, then let $a \to 0$

$$I(a) = \left(\frac{2}{3}(1)\ln 1 - \frac{4}{9}\right) - \left(\frac{2}{3}a^{3/2}\ln a - \frac{4}{9}a^{3/2}\right)$$

$$I(a) = -\frac{4}{9} - \frac{2}{3}a^{3/2}\ln a + \frac{4}{9}a^{3/2}$$

As $a \to 0$, $a^{3/2}\ln a \to 0$ and $a^{3/2} \to 0$

$$\boxed{I = -\frac{4}{9}}$$

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9 a. Use the substitution $u = e^x$ to show that

$$\int \frac{e^x}{e^{2x} + 9}\,dx = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right) + c$$

b. Hence find the exact value of $\int_{\ln 3}^{\infty} \frac{e^x}{e^{2x} + 9}\,dx$

(6 marks)

a) Substitute $u = e^x$

$$\frac{du}{dx} = e^x, \text{ so } e^x\,dx = du$$

$$e^{2x} = u^2$$

$$\int \frac{e^x}{(e^{2x} + 9)}\,dx = \int \frac{1}{(u^2 + 9)}\,du$$

$$\int \frac{1}{(u^2 + 9)}\,du = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + c$$

$$\boxed{\int \frac{e^x}{(e^{2x} + 9)}\,dx = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right) + c}$$

b) Integrate from ln 3 to t

$$I(t) = \frac{1}{3}\arctan\left(\frac{e^t}{3}\right) - \frac{1}{3}\arctan\left(\frac{e^{\ln 3}}{3}\right)$$

$$I(t) = \frac{1}{3}\arctan\left(\frac{e^t}{3}\right) - \frac{1}{3}\arctan 1$$

As $t \to \infty$, $\frac{e^t}{3} \to \infty$, so $\arctan\left(\frac{e^t}{3}\right) \to \frac{\pi}{2}$

$$I = \frac{1}{3}\left(\frac{\pi}{2}\right) - \frac{1}{3}\left(\frac{\pi}{4}\right)$$

$$\boxed{I = \frac{\pi}{12}}$$

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10 k is a positive constant.

a. Show that, for $k > 1$, $\int_1^\infty x^{-k}\,dx = \frac{1}{k-1}$

b. Explain, with working, what happens to $\int_1^\infty x^{-k}\,dx$ when $k = 1$ and when $0 < k < 1$

c. Find the value of k for which $\int_1^\infty x^{-k}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx$

(8 marks)

a) For $k \neq 1$

$$\int x^{-k}\,dx = \frac{x^{1-k}}{(1-k)} + c$$

Integrate from 1 to t

$$I(t) = \frac{(t^{1-k} - 1)}{(1-k)}$$

$k > 1$ means $1 - k < 0$, so $t^{1-k} = \frac{1}{t^{k-1}} \to 0$

$$I = \frac{(0-1)}{(1-k)}$$

$$\boxed{I = \frac{1}{(k-1)}}$$

b) $k = 1$

$$\int x^{-1}\,dx = \ln t - \ln 1 = \ln t$$

$\ln t \to \infty$ as $t \to \infty$, so divergent

$0 < k < 1$ means $1 - k > 0$

$t^{1-k} \to \infty$, so $I(t) \to \infty$: divergent

Divergent for $k = 1$ and for $0 < k < 1$

c) $x^{-1/2}$ is not defined at $x = 0$, so integrate from a to 1

$$\int x^{-1/2}\,dx = 2x^{1/2} + c$$

$$J(a) = 2 - 2\sqrt{a}$$

As $a \to 0$, $J \to 2$

$$\frac{1}{(k-1)} = 2$$

$$k - 1 = \frac{1}{2}$$

$$\boxed{k = \frac{3}{2}}$$

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11 A gas cylinder develops a leak. The rate at which gas escapes, R litres per hour, is modelled by

$R = 40te^{-\frac{t}{2}} \quad t \geq 0$

where t is the time in hours after the leak starts.

a. Show that, according to the model, the volume of gas that escapes in the first T hours is

$160 - (80T + 160)e^{-\frac{T}{2}}$ litres

b. Use an improper integral to find the total volume of gas that the model predicts will escape if the leak is never stopped. You must state clearly any limits you use.

c. Find the percentage of this total that escapes in the first 4 hours.

d. The cylinder held 120 litres of gas when the leak started. Comment on the model in the light of your answer to part (b).

(9 marks)

a) Integration by parts: $u = 40t$, $\frac{dv}{dt} = e^{-t/2}$

$$\frac{du}{dt} = 40$$

$$v = -2e^{-t/2}$$

$$\int 40te^{-t/2}\,dt = -80te^{-t/2} + \int 80e^{-t/2}\,dt$$

$$\int 40te^{-t/2}\,dt = -80te^{-t/2} - 160e^{-t/2} + c$$

Integrate from 0 to T

$$V(T) = (-80Te^{-T/2} - 160e^{-T/2}) - (0 - 160)$$

$$\boxed{V(T) = 160 - (80T + 160)e^{-T/2}}$$

b) Let $T \to \infty$

$Te^{-T/2} \to 0$ because the exponential dominates

$e^{-T/2} \to 0$

$$\boxed{\text{Total} = 160 \text{ Litres}}$$

c) $T = 4$

$$V(4) = 160 - 480e^{-2}$$

$$V(4) = 95.04\ldots$$

$$\text{percentage} = \frac{95.04}{160} \times 100$$

$$\boxed{\text{percentage} = 59.4\%}$$

d) Compare with the 120 litres in the cylinder

160 > 120, so the model predicts more gas escapes than there was

The model is not valid for large t; it must break down before the cylinder empties

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12 The curve C has equation

$$y = \frac{3}{2x + 1} \quad x \geq 0$$

The region R is bounded by C, the x-axis and the y-axis, and extends without limit in the positive x direction.

a. Show that the area of R is not finite.

b. The region R is rotated through 2π radians about the x-axis. Show that the volume of the solid formed is finite, and find its exact value.

c. The part of R with $0 \leq x \leq k$ is rotated through 2π radians about the x-axis. The volume of this solid is 99% of the volume found in part (b). Find the value of k.

(10 marks)

a) Area from 0 to t, then let $t \to \infty$

$$\int \frac{3}{(2x+1)}\,dx = \frac{3}{2}\ln(2x+1) + c$$

$$A(t) = \frac{3}{2}\ln(2t+1) - \frac{3}{2}\ln 1$$

$$A(t) = \frac{3}{2}\ln(2t+1)$$

As $t \to \infty$, $\ln(2t+1) \to \infty$

No finite limit, so the area is not finite

b) $V = \pi\int y^2\,dx$

$$y^2 = \frac{9}{(2x+1)^2}$$

$$\int 9(2x+1)^{-2}\,dx = -\frac{9}{2(2x+1)} + c$$

Integrate from 0 to t

$$V(t) = \pi\left(-\frac{9}{2(2t+1)} + \frac{9}{2}\right)$$

As $t \to \infty$, $\frac{9}{2(2t+1)} \to 0$

$V = \frac{9\pi}{2}$, which is finite

c) Volume from 0 to k

$$V(k) = \frac{9\pi}{2}\left(1 - \frac{1}{(2k+1)}\right)$$

$$\frac{9\pi}{2}\left(1 - \frac{1}{(2k+1)}\right) = 0.99 \times \frac{9\pi}{2}$$

$$\frac{1}{(2k+1)} = 0.01$$

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$2k + 1 = 100$

$\boxed{k = 49.5}$

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