Inverse trigonometric functions in calculus

Worked solutions

Further Maths topic practice

This topic online

1 Differentiate with respect to x

a. arcsin $(4x)$

b. arctan (x^3)

(3 marks)

Chain rule with the standard derivatives

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{(1-u^2)}} \times \frac{du}{dx}$$

$$\frac{d}{dx}(\arctan u) = \frac{1}{(1+u^2)} \times \frac{du}{dx}$$

a) $u = 4x$

$$\frac{d}{dx}(\arcsin 4x) = \frac{1}{\sqrt{(1-(4x)^2)}} \times 4$$

$$\boxed{\frac{d}{dx}(\arcsin 4x) = \frac{4}{\sqrt{(1-16x^2)}}}$$

b) $u = x^3$

$$\frac{d}{dx}(\arctan x^3) = \frac{1}{(1+(x^3)^2)} \times 3x^2$$

$$\boxed{\frac{d}{dx}(\arctan x^3) = \frac{3x^2}{(1+x^6)}}$$

2 Find

a. $\int \frac{1}{25+x^2}\,dx$

b. $\int \frac{3}{\sqrt{(4-x^2)}}\,dx$

(3 marks)

a) $\int \frac{1}{(a^2+x^2)}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + c,\ a = 5$

$$\boxed{\int \frac{1}{(25+x^2)}\,dx = \frac{1}{5}\arctan(\frac{x}{5}) + c}$$

b) $\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \arcsin(\frac{x}{a}) + c,\ a = 2$

$$\boxed{\int \frac{3}{\sqrt{(4-x^2)}}\,dx = 3\arcsin(\frac{x}{2}) + c}$$

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3 a. Given that $y = \arccos x$, $-1 < x < 1$, show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - x^2)}}$$

b. Given that $y = \sin(\arccos 3x)$, $-\frac{1}{3} < x < \frac{1}{3}$, determine $\frac{dy}{dx}$, giving your answer in its simplest form.

(5 marks)

a) Rewrite in terms of cos

$$x = \cos y$$

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

Use $\sin^2 y + \cos^2 y = 1$

$$\sin^2 y = 1 - x^2$$

$0 < y < \pi$, so $\sin y > 0$: take the positive root

$$\sin y = \sqrt{(1 - x^2)}$$

$$\boxed{\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - x^2)}}}$$

b) Chain rule

$$\frac{dy}{dx} = \cos(\arccos 3x) \times \frac{d}{dx}(\arccos 3x)$$

$$\frac{dy}{dx} = 3x \times \left(-\frac{3}{\sqrt{(1 - 9x^2)}}\right)$$

$$\boxed{\frac{dy}{dx} = -\frac{9x}{\sqrt{(1 - 9x^2)}}}$$

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4 Find the exact value of

a. $\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{3 + 4x^2}\,dx$

b. $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{(4 - 9x^2)}}\,dx$

(6 marks)

a) Take out the coefficient of x^2

$$\frac{1}{(3 + 4x^2)} = \frac{1}{4} \times \frac{1}{(3/4 + x^2)}$$

$a = \frac{\sqrt{3}}{2}$, so $\frac{1}{a} = \frac{2}{\sqrt{3}}$

$$\int = \frac{1}{4} \times \frac{2}{\sqrt{3}} \arctan\left(\frac{2x}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}} \arctan\left(\frac{2x}{\sqrt{3}}\right)$$

Limits 0 to $\frac{\sqrt{3}}{2}$

$$\int = \frac{1}{2\sqrt{3}} \arctan 1 - 0$$

$$\int = \frac{\pi}{8\sqrt{3}}$$

$$\boxed{\int = \frac{\sqrt{3}\pi}{24}}$$

b) Take out the coefficient of x^2

$$\sqrt{(4 - 9x^2)} = 3\sqrt{\left(\frac{4}{9} - x^2\right)}$$

$$\int = \frac{1}{3} \arcsin\left(\frac{3x}{2}\right)$$

Limits 0 to $\frac{1}{3}$

$$\int = \frac{1}{3} \arcsin\left(\frac{1}{2}\right) - 0$$

$$\boxed{\int = \frac{\pi}{18}}$$

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5 $f(x) = (1 + x^2) \arctan x$

a. Find $f'(x)$.

b. Show that

$(1 + x^2) f''(x) - 2f(x) = kx$

where k is an integer to be found.

(6 marks)

a) Product rule

$$f'(x) = 2x \arctan x + (1 + x^2) \times \frac{1}{(1 + x^2)}$$

$$\boxed{f'(x) = 2x \arctan x + 1}$$

b) Differentiate again, product rule on $2x \arctan x$

$$f''(x) = 2\arctan x + \frac{2x}{(1 + x^2)}$$

$$(1 + x^2) f''(x) = 2(1 + x^2)\arctan x + 2x$$

$$(1 + x^2) f''(x) = 2f(x) + 2x$$

$$(1 + x^2) f''(x) - 2f(x) = 2x$$

$$\boxed{k = 2}$$

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6 Show that

$$\int_0^4 \frac{4x - 3}{x^2 + 16}\,dx = a \ln 2 + b\pi$$

where a and b are rational numbers to be found.

(6 marks)

Split the fraction

$$\frac{(4x - 3)}{(x^2 + 16)} = \frac{4x}{(x^2 + 16)} - \frac{3}{(x^2 + 16)}$$

First term: the numerator is 2 x the derivative of $x^2 + 16$

$$\int \frac{4x}{(x^2 + 16)}\,dx = 2\ln(x^2 + 16)$$

Second term: arctan with a = 4

$$\int \frac{3}{(x^2 + 16)}\,dx = \frac{3}{4}\arctan\left(\frac{x}{4}\right)$$

Limits 0 to 4

$$\int = (2\ln 32 - \frac{3}{4}\arctan 1) - (2\ln 16 - \frac{3}{4}\arctan 0)$$

$$\int = 2\ln\left(\frac{32}{16}\right) - \frac{3}{4} \times \frac{\pi}{4}$$

$$\int = 2\ln 2 - \frac{3\pi}{16}$$

$$\boxed{a = 2,\ b = -\frac{3}{16}}$$

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7 a. Write $8x - x^2 - 7$ in the form $a - (x - b)^2$, where a and b are integers.

b. Hence find the exact value of

$$\int_4^{\frac{11}{2}} \frac{1}{\sqrt{(8x - x^2 - 7)}}\,dx$$

(5 marks)

a) Complete the square

$$8x - x^2 - 7 = -(x^2 - 8x) - 7$$

$$8x - x^2 - 7 = -((x - 4)^2 - 16) - 7$$

$$\boxed{8x - x^2 - 7 = 9 - (x - 4)^2}$$

b) $\int \frac{1}{\sqrt{(a^2 - u^2)}}\,du = \arcsin\left(\frac{u}{a}\right)$, with $u = x - 4$, $a = 3$

$$\int = \arcsin\left(\frac{(x - 4)}{3}\right)$$

Limits 4 to $\frac{11}{2}$

$$\int = \arcsin\left(\frac{1}{2}\right) - \arcsin 0$$

$$\boxed{\int = \frac{\pi}{6}}$$

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8 $f(x) = \dfrac{3x^2 + 4x + 11}{(x + 1)(x^2 + 4)}$

a. Express $f(x)$ in partial fractions.

b. Hence show that

$\int_0^2 f(x)\,dx = p \ln q + r\pi$

where p, q and r are rational numbers to be found.

(8 marks)

a) Quadratic factor needs a linear numerator

$$\frac{(3x^2 + 4x + 11)}{(x+1)(x^2+4)} = \frac{A}{(x+1)} + \frac{(Bx + C)}{(x^2+4)}$$

$$3x^2 + 4x + 11 = A(x^2 + 4) + (Bx + C)(x + 1)$$

$$x = -1: 3 - 4 + 11 = 5A, \text{ so } A = 2$$

Compare coefficients of x^2

$$3 = A + B, \text{ so } B = 1$$

Compare constant terms

$$11 = 4A + C, \text{ so } C = 3$$

$$f(x) = \frac{2}{(x+1)} + \frac{(x+3)}{(x^2+4)}$$

b) Split the second fraction

$$\frac{(x+3)}{(x^2+4)} = \frac{x}{(x^2+4)} + \frac{3}{(x^2+4)}$$

$$\int f(x)\,dx = 2\ln(x+1) + \frac{1}{2}\ln(x^2+4) + \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c$$

Limits 0 to 2

$$\int = (2\ln 3 + \frac{1}{2}\ln 8 + \frac{3}{2}\arctan 1) - (2\ln 1 + \frac{1}{2}\ln 4 + 0)$$

$$\int = 2\ln 3 + \frac{1}{2}\ln 2 + \frac{3\pi}{8}$$

$$2\ln 3 + \frac{1}{2}\ln 2 = \frac{1}{2}\ln(81 \times 2)$$

$$\int = \frac{1}{2}\ln 162 + \frac{3\pi}{8}, \text{ so } p = \frac{1}{2}, q = 162, r = \frac{3}{8}$$

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9 The curve C has equation

$y = \arctan(2x) - x$

a. Find the exact coordinates of the stationary points of C.

b. Determine the nature of each stationary point.

(7 marks)

a) Chain rule on arctan 2x

$$\frac{dy}{dx} = \frac{2}{(1+4x^2)} - 1$$

Stationary points: $\frac{dy}{dx} = 0$

$$\frac{2}{(1+4x^2)} = 1$$

$$4x^2 = 1$$

$$x = \pm\frac{1}{2}$$

$$x = \frac{1}{2}: y = \arctan 1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$$

$$x = -\frac{1}{2}: y = \arctan(-1) + \frac{1}{2} = \frac{1}{2} - \frac{\pi}{4}$$

$\left(\frac{1}{2}, \frac{\pi}{4} - \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2} - \frac{\pi}{4}\right)$

b) Second derivative

$$\frac{dy}{dx} = 2(1+4x^2)^{-1} - 1$$

$$\frac{d^2y}{dx^2} = -\frac{16x}{(1+4x^2)^2}$$

$$x = \frac{1}{2}: \frac{d^2y}{dx^2} = -\frac{8}{4} = -2 < 0$$

$$x = -\frac{1}{2}: \frac{d^2y}{dx^2} = \frac{8}{4} = 2 > 0$$

$\left(\frac{1}{2}, \frac{\pi}{4} - \frac{1}{2}\right)$ is a maximum; $\left(-\frac{1}{2}, \frac{1}{2} - \frac{\pi}{4}\right)$ is a minimum

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10 The curve C has equation

$y = x \arcsin x \quad -1 < x < 1$

The tangent to C at the point where $x = \frac{1}{2}$ meets the y-axis at the point P.

Show that the coordinates of P are $(0, k\sqrt{3})$, where k is a rational number to be found. (6 marks)

Product rule

$$\frac{dy}{dx} = \arcsin x + \frac{x}{\sqrt{(1 - x^2)}}$$

At $x = \frac{1}{2}$

$$y = \frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$$

$$\frac{dy}{dx} = \frac{\pi}{6} + (1/2)/(\sqrt{3}/2)$$

$$\frac{dy}{dx} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$$

Tangent: $y - y1 = m(x - x1)$

$$y - \frac{\pi}{12} = \left(\frac{\pi}{6} + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{2}\right)$$

Put $x = 0$

$$y = \frac{\pi}{12} - \frac{\pi}{12} - \frac{1}{2\sqrt{3}}$$

$$y = -\frac{1}{2\sqrt{3}} = -\frac{\sqrt{3}}{6}$$

$$P = \left(0, -\frac{1}{6}\sqrt{3}\right), \text{ so } k = -\frac{1}{6}$$

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11 The curve C has equation

$$y = \frac{e^x}{\sqrt{(9 - e^{2x})}} \quad x < \ln 3$$

a. Use the substitution $u = e^x$ to show that $\int y \, dx = \arcsin\left(\frac{e^x}{3}\right) + c$

b. Find the exact area of the region bounded by C, the x-axis and the lines $x = \ln\frac{3}{2}$ and $x = \ln\frac{3\sqrt{2}}{2}$

c. Prove that C has no stationary points.

(9 marks)

a) $u = e^x$

$$\frac{du}{dx} = e^x, \text{ so } e^x\,dx = du$$

$$e^{2x} = u^2$$

$$\int \frac{e^x}{\sqrt{(9 - e^{2x})}}\,dx = \int \frac{1}{\sqrt{(9 - u^2)}}\,du$$

$$\int \frac{1}{\sqrt{(9 - u^2)}}\,du = \arcsin\left(\frac{u}{3}\right) + c$$

$$\boxed{\int y\,dx = \arcsin\left(\frac{e^x}{3}\right) + c}$$

b) $y > 0$, so area $= \int y\,dx$

$$x = \ln\left(\frac{3\sqrt{2}}{2}\right): \frac{e^x}{3} = \frac{\sqrt{2}}{2}$$

$$x = \ln\left(\frac{3}{2}\right): \frac{e^x}{3} = \frac{1}{2}$$

$$\text{area} = \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$

$$\text{area} = \frac{\pi}{4} - \frac{\pi}{6}$$

$$\boxed{\text{area} = \frac{\pi}{12}}$$

c) Write $y = e^x(9 - e^{2x})^{-1/2}\}$ and use the product rule

$$\frac{dy}{dx} = e^x(9 - e^{2x})^{-1/2}\} + e^x \times \left(-\frac{1}{2}\right)(9 - e^{2x})^{-3/2}\} \times (-2e^{2x})$$

$$\frac{dy}{dx} = e^x(9 - e^{2x})^{-1/2}\} + e^{3x}(9 - e^{2x})^{-3/2}\}$$

Take out the common factor $e^x(9 - e^{2x})^{-3/2}\}$

$$\frac{dy}{dx} = e^x(9 - e^{2x})^{-3/2}\}(9 - e^{2x} + e^{2x})$$

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$$\frac{dy}{dx} = \frac{9e^x}{(9-e^{2x})^{3/2}}$$

$e^x > 0$, and $9 - e^{2x} > 0$ for $x < \ln 3$

$\frac{dy}{dx} > 0$ for all $x < \ln 3$, so it is never 0: no stationary points

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12 A mural is painted on a vertical wall. The bottom of the mural is 2 m above a visitor's eye level and the top of the mural is 5 m above the visitor's eye level. The visitor stands on level ground at a horizontal distance x metres from the wall.

The angle subtended at the visitor's eye by the mural is θ radians, where

$\theta = \arctan\left(\frac{5}{x}\right) - \arctan\left(\frac{2}{x}\right) \quad x > 0$

a. Show that $\frac{d\theta}{dx} = \frac{2}{x^2 + 4} - \frac{5}{x^2 + 25}$

b. Find the exact value of x for which $\frac{d\theta}{dx} = 0$

c. Show that this value of x gives a maximum value of θ, and find this maximum angle in degrees to 1 decimal place.

d. Give one limitation of the model.

(10 marks)

a) Chain rule with $u = \frac{5}{x}$, $\frac{du}{dx} = -\frac{5}{x^2}$

$$\frac{d}{dx}\arctan\left(\frac{5}{x}\right) = \frac{1}{(1 + 25/x^2)} \times \left(-\frac{5}{x^2}\right)$$

Multiply top and bottom by x^2

$$\frac{d}{dx}\arctan\left(\frac{5}{x}\right) = -\frac{5}{(x^2 + 25)}$$

$$\frac{d}{dx}\arctan\left(\frac{2}{x}\right) = -\frac{2}{(x^2 + 4)}$$

$$\frac{d\theta}{dx} = -\frac{5}{(x^2 + 25)} + \frac{2}{(x^2 + 4)}$$

$$\boxed{\frac{d\theta}{dx} = \frac{2}{(x^2 + 4)} - \frac{5}{(x^2 + 25)}}$$

b) Set $\frac{d\theta}{dx} = 0$

$$2(x^2 + 25) = 5(x^2 + 4)$$

$$2x^2 + 50 = 5x^2 + 20$$

$$3x^2 = 30$$

$$\boxed{x = \sqrt{10} \ (x > 0)}$$

c) Sign of $\frac{d\theta}{dx}$ either side of $\sqrt{10} = 3.16...$

$x = 3$: $\frac{d\theta}{dx} = \frac{2}{13} - \frac{5}{34} = 0.0068 > 0$

$x = 3.3$: $\frac{d\theta}{dx} = \frac{2}{14.89} - \frac{5}{35.89} = -0.0050 < 0$

+ then −, so θ has a maximum

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$$\theta = \arctan\left(\frac{5}{\sqrt{10}}\right) - \arctan\left(\frac{2}{\sqrt{10}}\right)$$

$$\theta = 1.0068... - 0.5639... = 0.4429... \text{ rad}$$

$$\theta = 0.4429... \times \frac{180}{\pi}$$

$$\boxed{\theta = 25.4^\circ}$$

d)

Visitors have different eye levels, and the eye is modelled as a single point

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