Mean value of a function

Worked solutions

Further Maths topic practice

This topic online

1 Find the mean value of $f(x) = 3x^2 - 2x + 1$ over the interval [1, 4]. (3 marks)

Mean value = $\frac{1}{(b-a)}\int f(x)\,dx$, x from a to b

$\int (3x^2 - 2x + 1)\,dx = x^3 - x^2 + x + c$

Limits 1 to 4

$\int = (64 - 16 + 4) - (1 - 1 + 1)$

$\int = 51$

mean $= \frac{1}{(4-1)} \times 51$

mean = 17

2 Show that the mean value of $f(x) = \frac{4}{2x+1}$ over the interval [0, 4] is ln 3 (3 marks)

$\int \frac{4}{(2x+1)}\,dx = 2\ln(2x+1) + c$

Limits 0 to 4

$\int = 2\ln 9 - 2\ln 1$

$\int = 2\ln 3^2 = 4\ln 3$

mean $= \frac{1}{4} \times 4\ln 3$

mean = ln 3

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3 Find the exact mean value of $f(x) = \sin^2 x$ over the interval $[0, \frac{\pi}{3}]$ (4 marks)

Use $\cos 2x = 1 - 2\sin^2 x$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4}\sin 2x + c$$

Limits 0 to $\frac{\pi}{3}$

$$\int = \frac{\pi}{6} - \frac{1}{4}\sin\left(\frac{2\pi}{3}\right)$$

$$\int = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

$$\text{mean} = \frac{3}{\pi}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

$$\boxed{\text{mean} = \frac{1}{2} - \frac{3\sqrt{3}}{8\pi}}$$

4 $f(x) = xe^x$

a. Find the exact mean value of $f(x)$ over the interval $[0, 2]$.

b. Hence write down the exact mean value of $3f(x) - 4$ over the interval $[0, 2]$. (5 marks)

a) Integration by parts: $u = x$, $\frac{dv}{dx} = e^x$

$$\int xe^x \, dx = xe^x - \int e^x \, dx$$

$$\int xe^x \, dx = (x - 1)e^x + c$$

Limits 0 to 2

$$\int = (1)e^2 - (-1)e^0$$

$$\int = e^2 + 1$$

$$\boxed{\text{mean} = \frac{(e^2 + 1)}{2}}$$

b) Mean of $af(x) + b = a \times (\text{mean of } f) + b$

$$\text{mean} = 3 \times \frac{(e^2 + 1)}{2} - 4$$

$$\boxed{\text{mean} = \frac{(3e^2 - 5)}{2}}$$

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5 $f(x) = \dfrac{1}{x^2 - 4x + 13}$

a. Write $x^2 - 4x + 13$ in the form $(x + p)^2 + q$, where p and q are integers.

b. Hence find $\int f(x)\,dx$

c. Find the exact mean value of $f(x)$ over the interval $[2, 5]$.

(6 marks)

a) Complete the square

$$x^2 - 4x + 13 = (x - 2)^2 + 9$$

b) Standard integral $\int \frac{1}{(u^2 + a^2)}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)$, with $a = 3$

$$\int f(x)\,dx = \frac{1}{3}\arctan\left(\frac{(x - 2)}{3}\right) + c$$

c) Limits 2 to 5

$$\int = \frac{1}{3}\arctan 1 - \frac{1}{3}\arctan 0$$

$$\int = \frac{1}{3} \times \frac{\pi}{4} = \frac{\pi}{12}$$

$$\text{mean} = \frac{1}{(5 - 2)} \times \frac{\pi}{12}$$

$$\text{mean} = \frac{\pi}{36}$$

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6 Find the exact mean value of

$f(x) = \dfrac{1}{\sqrt{(16 - 9x^2)}}$

over the interval $[0, \frac{2}{3}]$

(5 marks)

Take out the coefficient of x^2

$$\sqrt{(16 - 9x^2)} = 3\sqrt{\left(\frac{16}{9} - x^2\right)}$$

Standard integral $\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \arcsin\left(\frac{x}{a}\right)$, with $a = \frac{4}{3}$

$$\int f(x)\,dx = \frac{1}{3}\arcsin\left(\frac{3x}{4}\right) + c$$

Limits 0 to $\frac{2}{3}$

$$\int = \frac{1}{3}\arcsin\left(\frac{1}{2}\right) - \frac{1}{3}\arcsin 0$$

$$\int = \frac{1}{3} \times \frac{\pi}{6} = \frac{\pi}{18}$$

$$\text{mean} = \frac{1}{(2/3)} \times \frac{\pi}{18} = \frac{3}{2} \times \frac{\pi}{18}$$

$$\boxed{\text{mean} = \frac{\pi}{12}}$$

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7 The function f is continuous and the mean value of f(x) over the interval [1, 5] is 7

a. Find the mean value of 3f(x) + 2 over the interval [1, 5].

b. Find the mean value of f(x − 2) over the interval [3, 7].

c. Find the mean value of f(2x) over the interval $[\frac{1}{2}, \frac{5}{2}]$.

d. Given also that the mean value of f(x) over the interval [5, 9] is 4, find the mean value of f(x) over the interval [1, 9].

(6 marks)

Mean value × length of interval = integral

$\int f(x)\,dx = 4 \times 7 = 28$, x from 1 to 5

a) Mean of $af(x) + b = a \times$ mean $+ b$

mean $= 3 \times 7 + 2$

mean = 23

b) A translation of the graph and of the interval by 2 to the right

mean = 7

c) Let $u = 2x$, so $dx = \frac{1}{2}du$; $x = \frac{1}{2}$ gives $u = 1$, $x = \frac{5}{2}$ gives $u = 5$

$\int f(2x)\,dx = \frac{1}{2}\int f(u)\,du = \frac{1}{2} \times 28 = 14$

$$\text{mean} = \frac{14}{(5/2 - 1/2)} = \frac{14}{2}$$

mean = 7

d) Add the two integrals

$\int f(x)\,dx = 4 \times 4 = 16$, x from 5 to 9

$\int f(x)\,dx = 28 + 16 = 44$, x from 1 to 9

$$\text{mean} = \frac{44}{8}$$

mean = 5.5

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8 $f(x) = \dfrac{1}{\sqrt{(x^2 + 6x + 13)}}$

a. Write $x^2 + 6x + 13$ in the form $(x + p)^2 + q$, where p and q are integers.

b. Hence, using a standard integral from the formula book, find the exact mean value of $f(x)$ over the interval $[-3, -\frac{3}{2}]$, giving your answer in the form $a \ln b$, where a and b are rational.

(6 marks)

a) Complete the square

$$x^2 + 6x + 13 = (x + 3)^2 + 4$$

b) Standard integral $\int \frac{1}{\sqrt{(u^2 + a^2)}}\, du = \operatorname{arsinh}\left(\frac{u}{a}\right)$, with $a = 2$

$$\int f(x)\, dx = \operatorname{arsinh}\left(\frac{(x + 3)}{2}\right) + c$$

Limits -3 to $-\frac{3}{2}$

$$\int = \operatorname{arsinh}\left(\frac{3}{4}\right) - \operatorname{arsinh} 0$$

$\operatorname{arsinh} u = \ln(u + \sqrt{(u^2 + 1)})$

$$\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{9}{16} + 1\right)}\right)$$

$$\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$$

$$\text{mean} = \frac{1}{(3/2)} \times \ln 2$$

$$\text{mean} = \frac{2}{3} \ln 2$$

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9 a. Use integration by parts to find $\int \arctan x \, dx$

b. Show that the mean value of $\arctan x$ over the interval $[0, \sqrt{3}]$ is

$\frac{\pi}{3} - \frac{\sqrt{3}}{3} \ln 2$

c. Hence write down the mean value of $\arctan(\frac{x}{2})$ over the interval $[0, 2\sqrt{3}]$. Give a reason for your answer.

(8 marks)

a) Integration by parts: $u = \arctan x$, $\frac{dv}{dx} = 1$

$$\frac{du}{dx} = \frac{1}{(1 + x^2)}$$

$$v = x$$

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{(1 + x^2)} \, dx$$

$$\boxed{\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1 + x^2) + c}$$

b) Limits 0 to $\sqrt{3}$

$$\int = \sqrt{3} \arctan \sqrt{3} - \frac{1}{2} \ln 4 - 0$$

$$\int = \frac{\sqrt{3}\pi}{3} - \ln 2$$

$$\text{mean} = \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}\pi}{3} - \ln 2 \right)$$

$$\text{mean} = \frac{\pi}{3} - \frac{\ln 2}{\sqrt{3}}$$

$$\boxed{\text{mean} = \frac{\pi}{3} - \frac{\sqrt{3}}{3} \ln 2}$$

c) Stretch, scale factor 2, parallel to the x-axis

The graph and the interval are both stretched; the heights are unchanged

$$\boxed{\text{mean} = \frac{\pi}{3} - \frac{\sqrt{3}}{3} \ln 2}$$

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10 After heavy rain, the rate of flow of water in a river, R m^3 s^{-1}, is modelled by

$R = 20 + 90te^{-\frac{t}{2}} \quad 0 \leq t \leq 10$

where t is the time in days after the rain starts.

a. Use calculus to find the mean rate of flow over the 10 days, giving your answer to 3 significant figures.

b. Hence estimate the total volume of water that flows past a point on the river during the 10 days. (1 day = 86 400 seconds.)

c. Measurements show that the actual mean rate of flow over the 10 days was 47 m^3 s^{-1}. Suggest how the model could be refined.

(8 marks)

a) Mean $= \frac{1}{10}\int R\,dt$, t from 0 to 10

Integration by parts: $u = 90t$, $\frac{dv}{dt} = e^{-t/2}$

$$\int 90te^{-t/2}\,dt = -180te^{-t/2} + \int 180e^{-t/2}\,dt$$

$$\int 90te^{-t/2}\,dt = -180te^{-t/2} - 360e^{-t/2} + c$$

Limits 0 to 10

$$\int 90te^{-t/2}\,dt = (-1800e^{-5} - 360e^{-5}) - (0 - 360)$$

$$\int 90te^{-t/2}\,dt = 360 - 2160e^{-5}$$

$$\int 20\,dt = 200$$

$$\text{mean} = \frac{1}{10}(560 - 2160e^{-5})$$

$$\text{mean} = 56 - 216e^{-5} = 54.545\ldots$$

mean = 54.5 m^3 s^{-1}

b) Volume = mean rate x time in seconds

$V = 54.545\ldots \times 10 \times 86400$

$V = 4.71 \times 10^7$ m^3

c) The model overestimates the mean

Reduce the 90 (or the base flow 20) so that the mean rate is 47

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11 The temperature, θ °C, inside a greenhouse is modelled by

$\theta = 14 - 5\cos\left(\frac{\pi t}{12}\right) \quad 0 \le t \le 24$

where t is the number of hours after midnight.

a. Use calculus to show that, according to the model, the mean temperature over the whole day is 14 °C.

b. Find, according to the model, the mean temperature between 6 am and 6 pm. Give your answer in °C to 3 significant figures.

c. The temperature in degrees Fahrenheit is $F = 1.8\theta + 32$. Find the mean temperature between 6 am and 6 pm in degrees Fahrenheit, explaining how you used your answer to part (b).

d. Give one reason why the mean temperature recorded inside the greenhouse on a particular day might differ from 14 °C.

(9 marks)

a)

$$\int \left(14 - 5\cos\left(\frac{\pi t}{12}\right)\right) dt = 14t - \frac{60}{\pi}\sin\left(\frac{\pi t}{12}\right) + c$$

Limits 0 to 24

$$\int = \left(336 - \frac{60}{\pi}\sin 2\pi\right) - (0 - 0)$$

$$\int = 336$$

$$\text{mean} = \frac{336}{24}$$

$$\boxed{\text{mean} = 14\ ^\circ\text{C}}$$

b) 6 am to 6 pm is $t = 6$ to $t = 18$

$$\int = \left(252 - \frac{60}{\pi}\sin\left(\frac{3\pi}{2}\right)\right) - \left(84 - \frac{60}{\pi}\sin\left(\frac{\pi}{2}\right)\right)$$

$$\int = \left(252 + \frac{60}{\pi}\right) - \left(84 - \frac{60}{\pi}\right)$$

$$\int = 168 + \frac{120}{\pi}$$

$$\text{mean} = \frac{1}{12}\left(168 + \frac{120}{\pi}\right)$$

$$\text{mean} = 14 + \frac{10}{\pi} = 17.183\ldots$$

$$\boxed{\text{mean} = 17.2\ ^\circ\text{C}}$$

c) Mean of $a\theta + b = a \times (\text{mean of } \theta) + b$

$$\text{mean of } F = 1.8 \times 17.183\ldots + 32$$

$$\boxed{\text{mean of } F = 62.9\ ^\circ\text{F}}$$

d)

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The weather (sunshine, cloud, season) changes the temperature, but the model is the same every day

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12 $f(x) = \dfrac{1}{x^2 + 2x + 5}$

a. Show that the mean value of $f(x)$ over the interval $[-1, 1]$ is $\frac{\pi}{16}$

b. Find the value of x in the interval $[-1, 1]$ for which $f(x)$ is equal to its mean value over this interval. Give your answer to 3 decimal places.

c. Show that $\int_{-1}^{\infty} f(x)\,dx$ converges and find its exact value.

d. Hence explain what happens to the mean value of $f(x)$ over the interval $[-1, k]$ as $k \to \infty$

(10 marks)

a) Complete the square

$$x^2 + 2x + 5 = (x+1)^2 + 4$$

$$\int \frac{1}{((x+1)^2 + 2^2)}\,dx = \frac{1}{2}\arctan\left(\frac{(x+1)}{2}\right) + c$$

Limits -1 to 1

$$\int = \frac{1}{2}\arctan 1 - \frac{1}{2}\arctan 0 = \frac{\pi}{8}$$

$$\text{mean} = \frac{1}{2} \times \frac{\pi}{8}$$

$$\boxed{\text{mean} = \frac{\pi}{16}}$$

b) Solve $f(x) = \frac{\pi}{16}$

$$(x+1)^2 + 4 = \frac{16}{\pi}$$

$$(x+1)^2 = \frac{16}{\pi} - 4 = 1.0929\ldots$$

$x \geq -1$, so take the positive root

$$x = -1 + \sqrt{1.0929\ldots}$$

$$\boxed{x = 0.045}$$

c) Integrate from -1 to t, then let $t \to \infty$

$$I(t) = \frac{1}{2}\arctan\left(\frac{(t+1)}{2}\right)$$

As $t \to \infty$, arctan $\to \frac{\pi}{2}$

$$\boxed{\int f(x)\,dx = \frac{\pi}{4}, \text{ so it converges}}$$

d) Mean over $[-1, k]$

$$\text{mean} = \frac{1}{(k+1)} \times \frac{1}{2}\arctan\left(\frac{(k+1)}{2}\right)$$

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The integral $\to \frac{\pi}{4}$ but $k + 1 \to \infty$

The mean value $\to 0$ as $k \to \infty$

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