Method of differences

Worked solutions

Further Maths topic practice

This topic online

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1

a. Express $\frac{1}{(2r-1)(2r+1)}$ in partial fractions.

b. Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(5 marks)

a)

$$\frac{1}{(2r-1)(2r+1)} = \frac{A}{(2r-1)} + \frac{B}{(2r+1)}$$

$$1 = A(2r+1) + B(2r-1)$$

$r = \frac{1}{2}$: $1 = 2A$

$A = \frac{1}{2}$

$r = -\frac{1}{2}$: $1 = -2B$

$B = -\frac{1}{2}$

$$\frac{1}{(2r-1)(2r+1)} = \frac{1}{(2(2r-1))} - \frac{1}{(2(2r+1))}$$

b)

Write out the first and last rows

$r = 1$: $\frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$

$r = 2$: $\frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

$r = 3$: $\frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$

...

$r = n$: $\frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$

Everything cancels except the first and last fractions

$$\text{Sum} = \frac{1}{2}\left(1 - \frac{1}{(2n+1)}\right)$$

$$\text{Sum} = \frac{1}{2} \times \frac{2n}{(2n+1)}$$

$$\text{Sum} = \frac{n}{(2n+1)}$$

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2 $f(r) = r^2(r+1)^2$

a. Show that $f(r) - f(r-1) = 4r^3$

b. Hence use the method of differences to prove that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5 marks)

a)

$$f(r-1) = (r-1)^2r^2$$

$$f(r) - f(r-1) = r^2(r+1)^2 - r^2(r-1)^2$$

$$f(r) - f(r-1) = r^2[(r+1)^2 - (r-1)^2]$$

$$f(r) - f(r-1) = r^2[(r^2+2r+1) - (r^2-2r+1)]$$

$$f(r) - f(r-1) = r^2 \times 4r$$

$$\boxed{f(r) - f(r-1) = 4r^3}$$

b)

Sum both sides from $r = 1$ to n

$r = 1$: $f(1) - f(0)$

$r = 2$: $f(2) - f(1)$

$r = 3$: $f(3) - f(2)$

...

$r = n$: $f(n) - f(n-1)$

$$\sum(r = 1 \text{ to } n)\ 4r^3 = f(n) - f(0)$$

$$4\sum r^3 = n^2(n+1)^2 - 0$$

$$\boxed{\sum(r = 1 \text{ to } n)\ r^3 = \frac{1}{4}n^2(n+1)^2}$$

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3

a. Show that $\frac{1}{\sqrt{(r+1)}+\sqrt{r}} = \sqrt{(r+1)} - \sqrt{r}$

b. Hence show that $\sum_{r=1}^{n} \frac{1}{\sqrt{(r+1)}+\sqrt{r}} = \sqrt{(n+1)} - 1$

c. Find the value of n for which the sum in part (b) is equal to 20

d. Explain why the infinite series $\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \ldots$ does not have a finite sum.

(7 marks)

a)

Multiply top and bottom by $\sqrt{(r+1)} - \sqrt{r}$

$$\frac{1}{(\sqrt{(r+1)}+\sqrt{r})} = \frac{(\sqrt{(r+1)}-\sqrt{r})}{((r+1)-r)}$$

$$\boxed{\frac{1}{(\sqrt{(r+1)}+\sqrt{r})} = \sqrt{(r+1)} - \sqrt{r}}$$

b)

$r = 1$: $\sqrt{2} - \sqrt{1}$

$r = 2$: $\sqrt{3} - \sqrt{2}$

$r = 3$: $\sqrt{4} - \sqrt{3}$

...

$r = n$: $\sqrt{(n+1)} - \sqrt{n}$

Everything cancels except $\sqrt{(n+1)}$ and $-\sqrt{1}$

$$\boxed{\text{Sum} = \sqrt{(n+1)} - 1}$$

c)

$$\sqrt{(n+1)} - 1 = 20$$

$$\sqrt{(n+1)} = 21$$

$$n + 1 = 441$$

$$\boxed{n = 440}$$

d)

The sum of the first n terms is $\sqrt{(n+1)} - 1$

$\sqrt{(n+1)} - 1 \to \infty$ as $n \to \infty$

$$\boxed{\text{The partial sums grow without limit, so there is no finite sum}}$$

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4 a. Show that $(r + 1)! - r! = r \times r!$

b. Hence show that $\sum_{r=1}^{n} r \times r! = (n + 1)! - 1$

c. Find the smallest value of n for which $\sum_{r=1}^{n} r \times r! > 1\,000\,000$

(6 marks)

a)

$$(r+1)! = (r+1) \times r!$$

$$(r+1)! - r! = (r+1)r! - r!$$

$$(r+1)! - r! = r!(r+1-1)$$

$$\boxed{(r+1)! - r! = r \times r!}$$

b)

$r = 1$: $2! - 1!$

$r = 2$: $3! - 2!$

$r = 3$: $4! - 3!$

...

$r = n$: $(n+1)! - n!$

All the middle terms cancel

$$\Sigma(r=1 \text{ to } n)\ r \times r! = (n+1)! - 1!$$

$$\boxed{\Sigma(r=1 \text{ to } n)\ r \times r! = (n+1)! - 1}$$

c)

Need $(n+1)! - 1 > 1000000$

$9! - 1 = 362879$

$10! - 1 = 3628799$

$n + 1 = 9$ is too small, $n + 1 = 10$ works

$$\boxed{n = 9}$$

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5 Use the method of differences to show that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{4}{(r+1)(r+3)} = \frac{n(an+b)}{3(n+2)(n+3)}$$

where a and b are integers to be found. (6 marks)

Partial fractions

$$\frac{4}{((r+1)(r+3))} = \frac{A}{(r+1)} + \frac{B}{(r+3)}$$

$$4 = A(r+3) + B(r+1)$$

$r = -1$: $4 = 2A$

$A = 2$

$r = -3$: $4 = -2B$

$B = -2$

$$\frac{4}{((r+1)(r+3))} = \frac{2}{(r+1)} - \frac{2}{(r+3)}$$

Write out the rows: each fraction cancels with one two rows later

$r = 1$: $\frac{2}{2} - \frac{2}{4}$

$r = 2$: $\frac{2}{3} - \frac{2}{5}$

$r = 3$: $\frac{2}{4} - \frac{2}{6}$

...

$r = n-1$: $\frac{2}{n} - \frac{2}{(n+2)}$

$r = n$: $\frac{2}{(n+1)} - \frac{2}{(n+3)}$

Left over: the first two positive terms and the last two negative terms

$$\text{Sum} = 1 + \frac{2}{3} - \frac{2}{(n+2)} - \frac{2}{(n+3)}$$

$$\text{Sum} = \frac{5}{3} - \frac{(2(n+3)+2(n+2))}{((n+2)(n+3))}$$

$$\text{Sum} = \frac{5}{3} - \frac{(4n+10)}{((n+2)(n+3))}$$

$$\text{Sum} = \frac{(5(n+2)(n+3) - 3(4n+10))}{(3(n+2)(n+3))}$$

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$$\text{Sum} = \frac{(5n^2 + 25n + 30 - 12n - 30)}{(3(n+2)(n+3))}$$

$$\text{Sum} = \frac{(5n^2 + 13n)}{(3(n+2)(n+3))}$$

$$\text{Sum} = \frac{n(5n + 13)}{(3(n+2)(n+3))}$$

$a = 5, b = 13$

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6

a. Show that $\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$

b. Hence show that $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$

c. Write down the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$

d. Find the exact value of $\sum_{r=10}^{\infty} \frac{1}{r(r+1)(r+2)}$

(8 marks)

a)

$$\frac{1}{(r(r+1))} - \frac{1}{((r+1)(r+2))} = \frac{((r+2)-r)}{(r(r+1)(r+2))}$$

$$\boxed{\frac{1}{(r(r+1))} - \frac{1}{((r+1)(r+2))} = \frac{2}{(r(r+1)(r+2))}}$$

b)

$$\frac{1}{(r(r+1)(r+2))} = \frac{1}{2}\left(\frac{1}{(r(r+1))} - \frac{1}{((r+1)(r+2))}\right)$$

$r = 1$: $\frac{1}{2}\left(\frac{1}{(1 \times 2)} - \frac{1}{(2 \times 3)}\right)$

$r = 2$: $\frac{1}{2}\left(\frac{1}{(2 \times 3)} - \frac{1}{(3 \times 4)}\right)$

...

$r = n$: $\frac{1}{2}\left(\frac{1}{(n(n+1))} - \frac{1}{((n+1)(n+2))}\right)$

$$\text{Sum} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{((n+1)(n+2))}\right)$$

$$\boxed{\text{Sum} = \frac{1}{4} - \frac{1}{(2(n+1)(n+2))}}$$

c)

As $n \to \infty$, $\frac{1}{(2(n+1)(n+2))} \to 0$

$$\boxed{\text{Sum to infinity} = \frac{1}{4}}$$

d)

$$\Sigma(r = 10 \text{ to } \infty) = \Sigma(r = 1 \text{ to } \infty) - \Sigma(r = 1 \text{ to } 9)$$

$$\Sigma(r = 1 \text{ to } 9) = \frac{1}{4} - \frac{1}{(2 \times 10 \times 11)}$$

$$\Sigma(r = 10 \text{ to } \infty) = \frac{1}{4} - \left(\frac{1}{4} - \frac{1}{220}\right)$$

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$$\Sigma(r = 10 \text{ to } \infty) = \frac{1}{220}$$

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7

a. Show that $\ln(1 - \frac{1}{r^2}) = \ln(r - 1) + \ln(r + 1) - 2\ln r$

b. Hence use the method of differences to show that, for $n \geq 2$,

$\sum_{r=2}^{n} \ln(1 - \frac{1}{r^2}) = \ln(\frac{n+1}{2n})$

c. Find the exact value of $\sum_{r=11}^{40} \ln(1 - \frac{1}{r^2})$, giving your answer in the form $\ln(\frac{p}{q})$ where p and q are integers.

d. State the value of $\sum_{r=2}^{\infty} \ln(1 - \frac{1}{r^2})$

(7 marks)

a)

$$1 - \frac{1}{r^2} = \frac{(r^2 - 1)}{r^2} = \frac{(r - 1)(r + 1)}{r^2}$$

Take logs and use the laws of logarithms

$$\ln(1 - \frac{1}{r^2}) = \ln(r - 1) + \ln(r + 1) - \ln(r^2)$$

$$\boxed{\ln(1 - \frac{1}{r^2}) = \ln(r - 1) + \ln(r + 1) - 2\ln r}$$

b)

Group the terms as a difference

$$\ln(1 - \frac{1}{r^2}) = [\ln(r - 1) - \ln r] - [\ln r - \ln(r + 1)]$$

Let $g(r) = \ln(r - 1) - \ln r$, so each term is $g(r) - g(r + 1)$

$r = 2$: $g(2) - g(3)$

$r = 3$: $g(3) - g(4)$

...

$r = n$: $g(n) - g(n + 1)$

Sum $= g(2) - g(n + 1)$

Sum $= (\ln 1 - \ln 2) - (\ln n - \ln(n + 1))$

Sum $= \ln(n + 1) - \ln 2 - \ln n$

$$\boxed{\text{Sum} = \ln(\frac{(n+1)}{(2n)})}$$

c)

$$\Sigma(r = 11 \text{ to } 40) = \Sigma(r = 2 \text{ to } 40) - \Sigma(r = 2 \text{ to } 10)$$

$$\Sigma(r = 11 \text{ to } 40) = \ln(\frac{41}{80}) - \ln(\frac{11}{20})$$

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$$\Sigma(r = 11 \text{ to } 40) = \ln\left(\frac{41}{80} \times \frac{20}{11}\right)$$

$$\boxed{\Sigma(r = 11 \text{ to } 40) = \ln\left(\frac{41}{44}\right)}$$

d)

$\frac{(n+1)}{(2n)} = \frac{1}{2} + \frac{1}{(2n)}$, which tends to $\frac{1}{2}$

$$\boxed{\text{Sum to infinity} = \ln\left(\frac{1}{2}\right) = -\ln 2}$$

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8 a. Use the method of differences to show that, for $n \geq 2$,

$$\sum_{r=2}^{n} \frac{4}{r^2-1} = \frac{(n-1)(3n+2)}{n(n+1)}$$

b. Hence show that $\sum_{r=n+1}^{2n} \frac{4}{r^2-1} = \frac{4n^2+3n+1}{n(n+1)(2n+1)}$

(8 marks)

a)

$$\frac{4}{(r^2-1)} = \frac{A}{(r-1)} + \frac{B}{(r+1)}$$

$$4 = A(r+1) + B(r-1)$$

$r = 1$ gives $A = 2$, $r = -1$ gives $B = -2$

$$\frac{4}{(r^2-1)} = \frac{2}{(r-1)} - \frac{2}{(r+1)}$$

$r = 2$: $\frac{2}{1} - \frac{2}{3}$

$r = 3$: $\frac{2}{2} - \frac{2}{4}$

$r = 4$: $\frac{2}{3} - \frac{2}{5}$

...

$r = n-1$: $\frac{2}{(n-2)} - \frac{2}{n}$

$r = n$: $\frac{2}{(n-1)} - \frac{2}{(n+1)}$

Left over: $2 + 1$ from the start, $-\frac{2}{n} - \frac{2}{(n+1)}$ from the end

$$\text{Sum} = 3 - \frac{2}{n} - \frac{2}{(n+1)}$$

$$\text{Sum} = \frac{(3n(n+1) - 2(n+1) - 2n)}{(n(n+1))}$$

$$\text{Sum} = \frac{(3n^2 - n - 2)}{(n(n+1))}$$

$$\boxed{\text{Sum} = \frac{(n-1)(3n+2)}{(n(n+1))}}$$

b)

Let $S(m) = 3 - \frac{2}{m} - \frac{2}{(m+1)}$, the sum from $r = 2$ to m

$$\Sigma(r = n+1 \text{ to } 2n) = S(2n) - S(n)$$

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$$\Sigma(r = n+1 \text{ to } 2n) = \left(3 - \frac{2}{(2n)} - \frac{2}{(2n+1)}\right) - \left(3 - \frac{2}{n} - \frac{2}{(n+1)}\right)$$

$$\Sigma(r = n+1 \text{ to } 2n) = \frac{1}{n} + \frac{2}{(n+1)} - \frac{2}{(2n+1)}$$

$$\Sigma(r = n+1 \text{ to } 2n) = \frac{((n+1)(2n+1) + 2n(2n+1) - 2n(n+1))}{(n(n+1)(2n+1))}$$

$$\Sigma(r = n+1 \text{ to } 2n) = \frac{(2n^2 + 3n + 1 + 4n^2 + 2n - 2n^2 - 2n)}{(n(n+1)(2n+1))}$$

$$\boxed{\Sigma(r = n+1 \text{ to } 2n) = \frac{(4n^2 + 3n + 1)}{(n(n+1)(2n+1))}}$$

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9

a. Express $\frac{3r + 4}{r(r + 1)(r + 2)}$ in partial fractions.

b. Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{3r + 4}{r(r + 1)(r + 2)} = \frac{n(5n + 9)}{2(n + 1)(n + 2)}$$

c. Find the sum to infinity of the series.

(9 marks)

a)

$$\frac{(3r + 4)}{(r(r + 1)(r + 2))} = \frac{A}{r} + \frac{B}{(r + 1)} + \frac{C}{(r + 2)}$$

$$3r + 4 = A(r + 1)(r + 2) + Br(r + 2) + Cr(r + 1)$$

$r = 0$: $4 = 2A$

$A = 2$

$r = -1$: $1 = -B$

$B = -1$

$r = -2$: $-2 = 2C$

$C = -1$

$$\boxed{\frac{2}{r} - \frac{1}{(r + 1)} - \frac{1}{(r + 2)}}$$

b)

Split $\frac{2}{r}$ into two halves that pair with the other fractions

$$\text{term} = \left(\frac{1}{r} - \frac{1}{(r + 1)}\right) + \left(\frac{1}{r} - \frac{1}{(r + 2)}\right)$$

First bracket: everything cancels except the first and last

$$\sum\left(\frac{1}{r} - \frac{1}{(r + 1)}\right) = 1 - \frac{1}{(n + 1)}$$

Second bracket: each fraction cancels with one two rows later

$r = 1$: $1 - \frac{1}{3}$

$r = 2$: $\frac{1}{2} - \frac{1}{4}$

$r = 3$: $\frac{1}{3} - \frac{1}{5}$

...

$r = n - 1$: $\frac{1}{(n - 1)} - \frac{1}{(n + 1)}$

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$r = n: \frac{1}{n} - \frac{1}{(n+2)}$

$$\sum\left(\frac{1}{r} - \frac{1}{(r+2)}\right) = 1 + \frac{1}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

$$\text{Sum} = \frac{5}{2} - \frac{2}{(n+1)} - \frac{1}{(n+2)}$$

$$\text{Sum} = \frac{(5(n+1)(n+2) - 4(n+2) - 2(n+1))}{(2(n+1)(n+2))}$$

$$\text{Sum} = \frac{(5n^2 + 15n + 10 - 4n - 8 - 2n - 2)}{(2(n+1)(n+2))}$$

$$\text{Sum} = \frac{(5n^2 + 9n)}{(2(n+1)(n+2))}$$

$$\boxed{\text{Sum} = \frac{n(5n+9)}{(2(n+1)(n+2))}}$$

c)

As $n \to \infty$, $\frac{2}{(n+1)}$ and $\frac{1}{(n+2)}$ both tend to 0

$$\boxed{\text{Sum to infinity} = \frac{5}{2}}$$

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10

a. Show that $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$

b. Hence find $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$, giving your answer in terms of n.

c. Find the exact value of $\sum_{r=10}^{\infty} \frac{2r+1}{r^2(r+1)^2}$

d. Find the smallest value of n for which $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} > 0.9999$

(8 marks)

a)

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{((r+1)^2 - r^2)}{(r^2(r+1)^2)}$$

$$\boxed{\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(2r+1)}{(r^2(r+1)^2)}}$$

b)

$r = 1$: $\frac{1}{1^2} - \frac{1}{2^2}$

$r = 2$: $\frac{1}{2^2} - \frac{1}{3^2}$

...

$r = n$: $\frac{1}{n^2} - \frac{1}{(n+1)^2}$

$$\boxed{\text{Sum} = 1 - \frac{1}{(n+1)^2}}$$

c)

As $n \to \infty$ the sum from $r = 1$ tends to 1

$$\Sigma(r = 10 \text{ to } \infty) = 1 - \Sigma(r = 1 \text{ to } 9)$$

$$\Sigma(r = 10 \text{ to } \infty) = 1 - \left(1 - \frac{1}{10^2}\right)$$

$$\boxed{\Sigma(r = 10 \text{ to } \infty) = \frac{1}{100}}$$

d)

$$1 - \frac{1}{(n+1)^2} > 0.9999$$

$$\frac{1}{(n+1)^2} < 0.0001$$

$$(n+1)^2 > 10000$$

$$n + 1 > 100$$

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$n = 99$ gives exactly 0.9999, which is not greater

$$\boxed{n = 100}$$

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11 a. Explain why, for any function f,

$\sum_{r=1}^{n} [f(r+1) - f(r-1)] = f(n+1) + f(n) - f(1) - f(0)$

b. Given that $f(r) = r^3$, show that $f(r+1) - f(r-1) = 6r^2 + 2$

c. Hence prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

d. Without using the formula in part (c), evaluate $\sum_{r=20}^{50} (6r^2 + 2)$

(10 marks)

a)

$r = 1$: $f(2) - f(0)$

$r = 2$: $f(3) - f(1)$

$r = 3$: $f(4) - f(2)$

...

$r = n-1$: $f(n) - f(n-2)$

$r = n$: $f(n+1) - f(n-1)$

Each $f(k)$ cancels with the term two rows later

$f(n+1)$ and $f(n)$ are never subtracted

$f(0)$ and $f(1)$ are never added

$$\text{Sum} = f(n+1) + f(n) - f(1) - f(0)$$

b)

$$(r+1)^3 = r^3 + 3r^2 + 3r + 1$$

$$(r-1)^3 = r^3 - 3r^2 + 3r - 1$$

$$(r+1)^3 - (r-1)^3 = 6r^2 + 2$$

$$f(r+1) - f(r-1) = 6r^2 + 2$$

c)

Use part (a) with $f(r) = r^3$

$$\Sigma(6r^2 + 2) = (n+1)^3 + n^3 - 1 - 0$$

$$6\Sigma r^2 + 2n = n^3 + 3n^2 + 3n + 1 + n^3 - 1$$

$$6\Sigma r^2 + 2n = 2n^3 + 3n^2 + 3n$$

$$6\Sigma r^2 = 2n^3 + 3n^2 + n$$

$$6\Sigma r^2 = n(2n^2 + 3n + 1)$$

$$6\Sigma r^2 = n(n+1)(2n+1)$$

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$$\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$$

d)

The same cancelling from r = 20 to 50 leaves $f(51) + f(50) - f(20) - f(19)$

$$\sum(r = 20 \text{ to } 50)(6r^2 + 2) = 51^3 + 50^3 - 20^3 - 19^3$$

$$\sum(r = 20 \text{ to } 50)(6r^2 + 2) = 132651 + 125000 - 8000 - 6859$$

$$\sum(r = 20 \text{ to } 50)(6r^2 + 2) = 242792$$

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12 A games company runs an online puzzle. The rth player to solve the puzzle wins a prize of £ $\frac{6000}{(r+1)(r+2)}$

a. Use the method of differences to show that the total amount paid to the first n players to solve the puzzle is £ $\frac{3000n}{n+2}$

b. Explain why the company will never pay out £3000 or more in total, however many players solve the puzzle.

c. Find the number of players who must solve the puzzle for the total paid out to first exceed £2900

d. Find the total amount won by the 21st to the 40th players inclusive, giving your answer to the nearest penny.

(9 marks)

a)

$$\frac{6000}{((r+1)(r+2))} = 6000\left(\frac{1}{(r+1)} - \frac{1}{(r+2)}\right)$$

$r = 1: 6000\left(\frac{1}{2} - \frac{1}{3}\right)$

$r = 2: 6000\left(\frac{1}{3} - \frac{1}{4}\right)$

...

$r = n: 6000\left(\frac{1}{(n+1)} - \frac{1}{(n+2)}\right)$

Everything cancels except the first and last fractions

$$\text{Total} = 6000\left(\frac{1}{2} - \frac{1}{(n+2)}\right)$$

$$\text{Total} = 6000 \times \frac{(n+2-2)}{(2(n+2))}$$

$$\boxed{\text{Total} = £\frac{3000n}{(n+2)}}$$

b)

$$\frac{3000n}{(n+2)} = \frac{(3000(n+2) - 6000)}{(n+2)} = 3000 - \frac{6000}{(n+2)}$$

$\frac{6000}{(n+2)} > 0$ for every n

The total is always less than £3000

c)

$$\frac{3000n}{(n+2)} > 2900$$

$3000n > 2900n + 5800$

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$100n > 5800$

$n > 58$

$n = 58$ gives exactly £2900, which does not exceed it

59 players

d)

$$\text{Total}(21 \text{ to } 40) = \text{Total}(40) - \text{Total}(20)$$

$$\text{Total}(21 \text{ to } 40) = \frac{(3000 \times 40)}{42} - \frac{(3000 \times 20)}{22}$$

$$\text{Total}(21 \text{ to } 40) = 2857.142\ldots - 2727.272\ldots$$

£129.87

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