Nth roots and roots of unity

Worked solutions

Further Maths topic practice

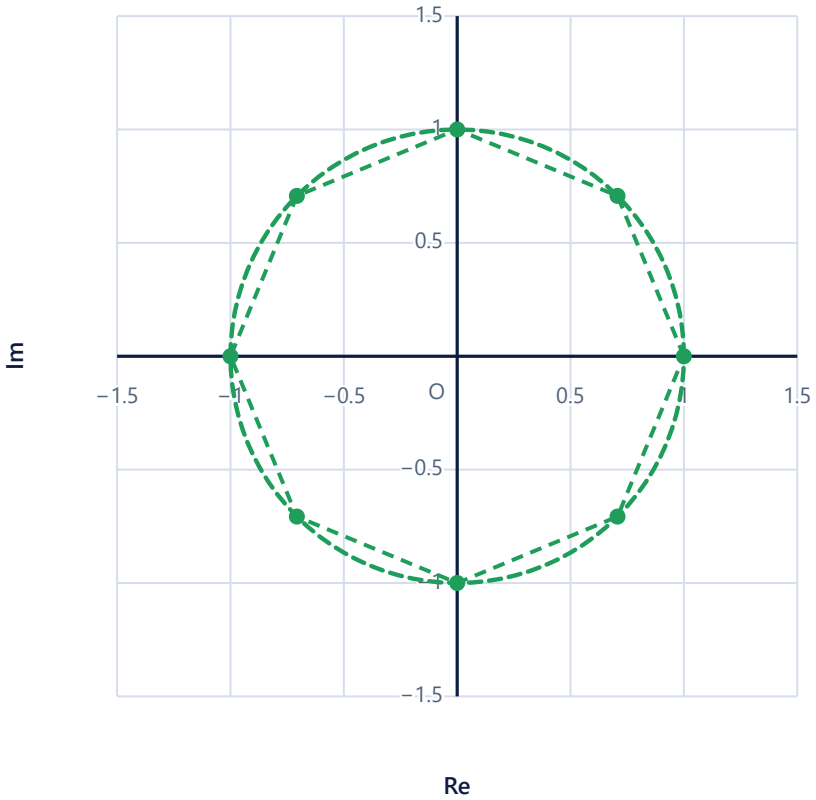
This topic online

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1 The equation $z^8 = 1$ has eight roots.

a. Solve the equation, giving each root in the form $a + bi$, where a and b are exact real numbers.

b. Show the eight roots on the Argand diagram below.



(4 marks)

a)

Write 1 in exponential form, adding $2k\pi$ to the argument

$$z^8 = e^{2k\pi i}$$

$$z = e^{k\pi i/4},\ k = 0, 1, 2, \ldots, 7$$

Convert each root using $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^0 = 1$$

$$e^{\pi i/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$e^{\pi i/2} = i$$

$$e^{3\pi i/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$e^{\pi i} = -1$$

$$e^{5\pi i/4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$e^{3\pi i/2} = -i$$

$$e^{7\pi i/4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

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$$z = \pm 1, \pm i, \pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$$

b)

Every root has modulus 1, so they lie on the circle $|z| = 1$

The arguments go up in steps of $\frac{\pi}{4}$, starting at 1

The roots are the vertices of a regular octagon

2 Solve the equation

$z^3 = 8i$

giving your answers in the form $a + bi$, where a and b are exact real numbers. (4 marks)

Write $8i$ in modulus-argument form

$$8i = 8e^{\pi i/2}$$

Add $2k\pi$ to the argument before taking the cube root

$$z^3 = 8e^{i(\pi/2 + 2k\pi)}$$

$$z = 2e^{i(\pi/6 + 2k\pi/3)}$$

$k = 0, 1, -1$

$$z = 2e^{\pi i/6}, 2e^{5\pi i/6}, 2e^{-\pi i/2}$$

Convert using $r(\cos\theta + i\sin\theta)$

$$2e^{\pi i/6} = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$2e^{5\pi i/6} = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$2e^{-\pi i/2} = 2(0 - i) = -2i$$

$$z = \sqrt{3} + i, -\sqrt{3} + i, -2i$$

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3 The complex number ω is a root of the equation $z^3 = 1$, where $\omega \neq 1$

a. Show that $1 + \omega + \omega^2 = 0$

b. Show that $(2 + \omega)(2 + \omega^2) = 3$

c. Find the value of $(1 + \omega^2)^6$

(6 marks)

a)

ω is a root of $z^3 - 1 = 0$

$z^3 - 1 = (z - 1)(z^2 + z + 1)$

$\omega \neq 1$, so $\omega - 1 \neq 0$

$\omega^2 + \omega + 1 = 0$

$\boxed{1 + \omega + \omega^2 = 0}$

b)

$(2 + \omega)(2 + \omega^2) = 4 + 2\omega^2 + 2\omega + \omega^3$

Use $\omega + \omega^2 = -1$ and $\omega^3 = 1$

$(2 + \omega)(2 + \omega^2) = 4 + 2(-1) + 1$

$\boxed{(2 + \omega)(2 + \omega^2) = 3}$

c)

From part (a), $1 + \omega^2 = -\omega$

$(1 + \omega^2)^6 = (-\omega)^6$

$(1 + \omega^2)^6 = \omega^6 = (\omega^3)^2$

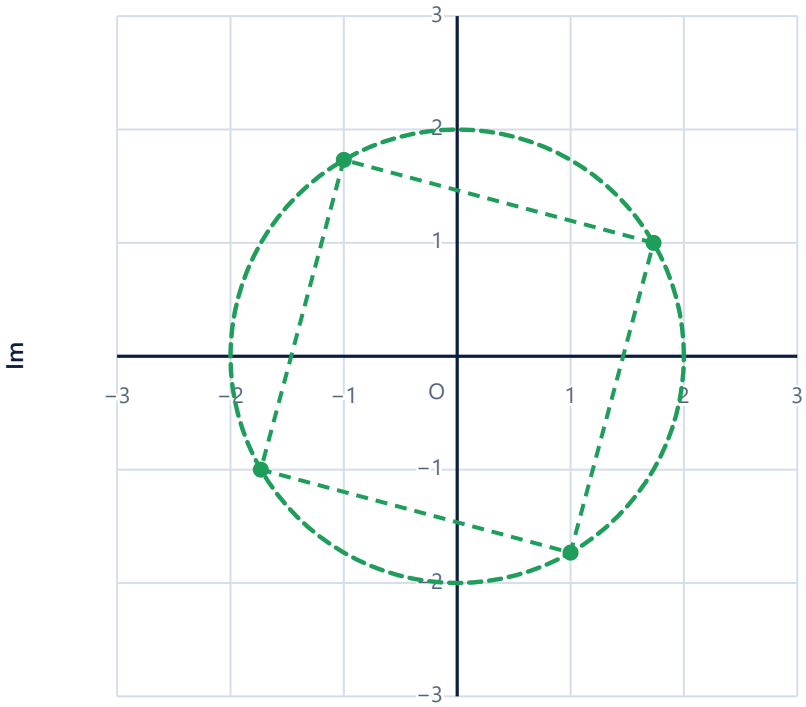
$\boxed{(1 + \omega^2)^6 = 1}$

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4

a. Solve the equation $z^4 = -8 + 8\sqrt{3}i$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$

b. Show the roots on the Argand diagram below and describe the shape they form.



(7 marks)

a)

Find the modulus and argument of $-8 + 8\sqrt{3}i$

$$\left|-8 + 8\sqrt{3}i\right| = \sqrt{(64 + 192)} = \sqrt{256} = 16$$

The point is in the second quadrant

$$\arg(-8 + 8\sqrt{3}i) = \pi - \arctan\left(\frac{8\sqrt{3}}{8}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Add $2k\pi$ to the argument before taking the fourth root

$$z^4 = 16e^{i(2\pi/3 + 2k\pi)}$$

The fourth root of 16 is 2

$$z = 2e^{i(\pi/6 + k\pi/2)}$$

$k = 0, 1, -1, -2$ keep the arguments in $-\pi < \theta \leq \pi$

$$z = 2e^{\pi i/6}, 2e^{2\pi i/3}, 2e^{-\pi i/3}, 2e^{-5\pi i/6}$$

b)

All four roots lie on the circle $|z| = 2$

The arguments are $\frac{\pi}{2}$ apart, starting at $\frac{\pi}{6}$

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The roots are the vertices of a square centred at O

5 a. Write down the roots of the equation $w^6 = 1$, giving your answers in the form $e^{i\theta}$, where $0 \le \theta < 2\pi$

b. Show that $(3e^{\pi i/6})^6 = -729$

c. Hence, or otherwise, solve the equation $z^6 + 729 = 0$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$

(7 marks)

a)

$w^6 = e^{2k\pi i}$

$w = e^{k\pi i/3}$, $k = 0, 1, 2, 3, 4, 5$

$w = 1, e^{\pi i/3}, e^{2\pi i/3}, e^{\pi i}, e^{4\pi i/3}, e^{5\pi i/3}$

b)

$(3e^{\pi i/6})^6 = 3^6 e^{6\pi i/6}$

$(3e^{\pi i/6})^6 = 729e^{\pi i}$

$e^{\pi i} = \cos\pi + i\sin\pi = -1$

$(3e^{\pi i/6})^6 = -729$

c)

$z^6 = -729 = (3e^{\pi i/6})^6$

So $z = 3e^{\pi i/6} \times w$, where w is a sixth root of unity

$z = 3e^{\pi i/6}e^{k\pi i/3}$

$z = 3e^{(2k+1)\pi i/6}$, $k = 0, 1, \ldots, 5$

$z = 3e^{\pi i/6}, 3e^{\pi i/2}, 3e^{5\pi i/6}, 3e^{7\pi i/6}, 3e^{3\pi i/2}, 3e^{11\pi i/6}$

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6 The complex number $w = -4\sqrt{2} + 4\sqrt{2}i$

a. Solve the equation $z^3 = w$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$

b. Write the root that lies in the first quadrant in the form $a + bi$, where a and b are exact.

The points representing the three roots form the vertices of a triangle T in an Argand diagram.

c. Find the exact area of T.

(9 marks)

a)

$$|w| = \sqrt{(32 + 32)} = \sqrt{64} = 8$$

w is in the second quadrant

$$\arg w = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$z^3 = 8e^{i(3\pi/4 + 2k\pi)}$$

$$z = 2e^{i(\pi/4 + 2k\pi/3)}$$

$k = 0, 1, -1$

$$\frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$$

$$\frac{\pi}{4} - \frac{2\pi}{3} = -\frac{5\pi}{12}$$

$$\boxed{z = 2e^{\pi i/4}, 2e^{11\pi i/12}, 2e^{-5\pi i/12}\}}$$

b)

$$2e^{\pi i/4} = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

$$2e^{\pi i/4} = 2(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$$

$$\boxed{\sqrt{2} + \sqrt{2}i}$$

c)

The roots are equally spaced on $|z| = 2$, so T is equilateral

Split T into three triangles from O, each with two sides 2 and angle $\frac{2\pi}{3}$

$$\text{Area of each} = \frac{1}{2} \times 2 \times 2 \times \sin\frac{2\pi}{3}$$

$$\text{Area of each} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\text{Area of T} = 3 \times \sqrt{3}$$

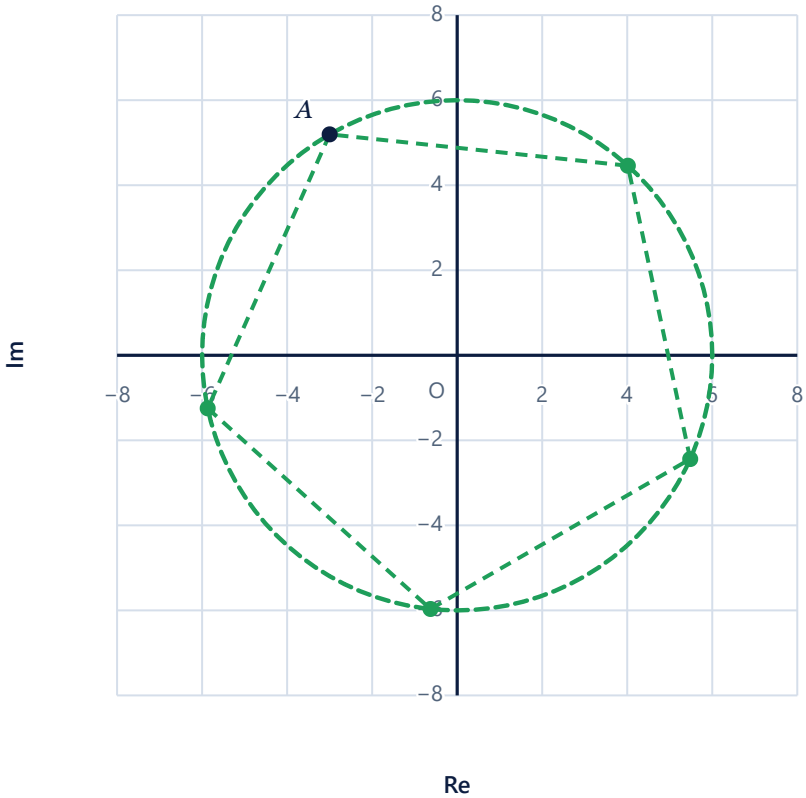
$$\boxed{\text{Area of T} = 3\sqrt{3}}$$

7 The point A represents the complex number $-3 + 3\sqrt{3}i$ in the Argand diagram below.

A is one vertex of a regular pentagon whose centre is the origin O.

a. Find the complex numbers represented by the other four vertices of the pentagon, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$

b. Find the area of the pentagon, giving your answer to 3 significant figures.



(7 marks)

a)

$$|A| = \sqrt{(9 + 27)} = \sqrt{36} = 6$$

$$\arg A = \pi - \arctan\left(\frac{3\sqrt{3}}{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Rotate A about O by multiples of $\frac{2\pi}{5}$

$$z = 6e^{i(2\pi/3 + 2k\pi/5)}$$

Subtract 2π from any argument greater than π

$$\frac{2\pi}{3} + \frac{2\pi}{5} = \frac{16\pi}{15}$$

$$\frac{16\pi}{15} - 2\pi = -\frac{14\pi}{15}$$

$$\frac{2\pi}{3} + \frac{4\pi}{5} = \frac{22\pi}{15}$$

$$\frac{22\pi}{15} - 2\pi = -\frac{8\pi}{15}$$

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$$\frac{2\pi}{3} + \frac{6\pi}{5} = \frac{28\pi}{15}$$

$$\frac{28\pi}{15} - 2\pi = -\frac{2\pi}{15}$$

$$\frac{2\pi}{3} + \frac{8\pi}{5} = \frac{34\pi}{15}$$

$$\frac{34\pi}{15} - 2\pi = \frac{4\pi}{15}$$

$$\boxed{6e^{4\pi i/15},\ 6e^{-2\pi i/15},\ 6e^{-8\pi i/15},\ 6e^{-14\pi i/15}}$$

b)

Five triangles from O, each with two sides 6 and angle $\frac{2\pi}{5}$

$$\text{Area} = 5 \times \frac{1}{2} \times 6 \times 6 \times \sin\frac{2\pi}{5}$$

$$\text{Area} = 90 \sin\frac{2\pi}{5}$$

$$\text{Area} = 85.595\ldots$$

$$\boxed{\text{Area} = 85.6 \text{ (3 s.f.)}}$$

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8 The points A, B and C are the vertices of an equilateral triangle in an Argand diagram. The centre of the triangle is the point G, which represents $2 + i$, and A represents $2 + 5i$.

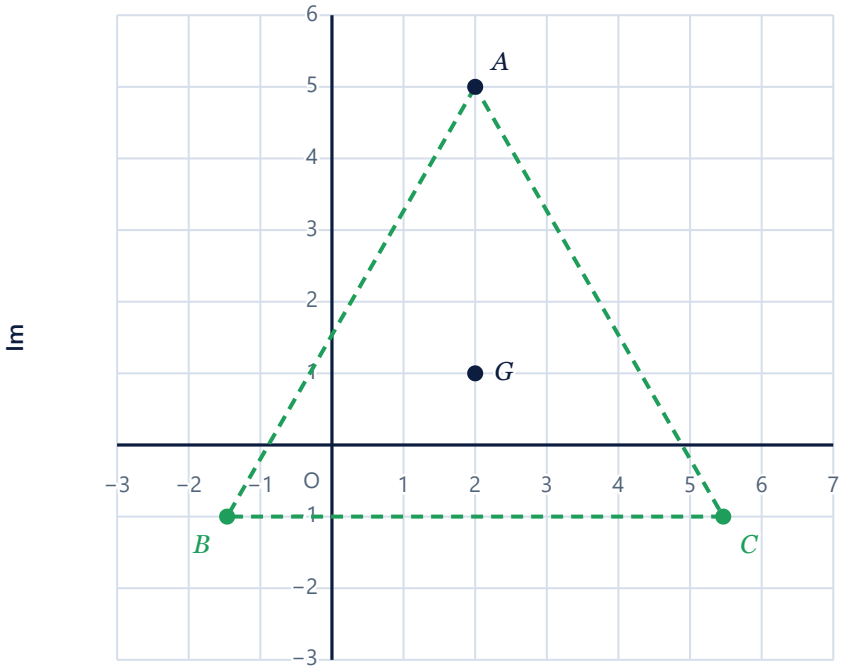
a. Write down the three roots of $z^3 = 1$ in the form $a + bi$, where a and b are exact.

The transformation $f(z) = pz + q$, where p and q are complex constants, maps the three roots of $z^3 = 1$ onto A, B and C, with the root 1 mapped to A.

b. Find p and q.

c. Hence find the complex numbers represented by B and C, giving your answers in exact form $a + bi$.

d. Find the exact area of triangle ABC.



(9 marks)

a)

$$z = e^{2k\pi i/3},\ k = 0, 1, -1$$

$$e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$e^{-2\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\boxed{z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

b)

The centre of the roots of unity is 0, which must map to G

$$f(0) = q = 2 + i$$

The root 1 maps to A

$$p + 2 + i = 2 + 5i$$

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$$p = 4i,\ q = 2 + i$$

c)

$$f\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 4i\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2 + i$$

$$f\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2i - 2\sqrt{3} + 2 + i$$

$$f\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = (2 - 2\sqrt{3}) - i$$

$$f\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -2i + 2\sqrt{3} + 2 + i$$

$$f\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = (2 + 2\sqrt{3}) - i$$

$$B = (2 - 2\sqrt{3}) - i,\ C = (2 + 2\sqrt{3}) - i$$

d)

$$BC = (2 + 2\sqrt{3}) - (2 - 2\sqrt{3}) = 4\sqrt{3}$$

$$\text{Area} = \frac{1}{2} \times (4\sqrt{3})^2 \times \sin 60^\circ$$

$$\text{Area} = \frac{1}{2} \times 48 \times \frac{\sqrt{3}}{2}$$

$$\text{Area} = 12\sqrt{3}$$

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9

a. Solve the equation $z^5 = 1$, giving your answers in the form $e^{i\theta}$, where $-\pi < \theta \leq \pi$

b. Explain why the sum of the five roots is 0

c. Hence show that $\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$

d. Using the identity $\cos 2A \equiv 2\cos^2 A - 1$, show that $\cos \frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$

(9 marks)

a)

$z^5 = e^{2k\pi i}$

$z = e^{2k\pi i/5}, k = 0, \pm 1, \pm 2$

$\boxed{z = 1, e^{2\pi i/5}, e^{-2\pi i/5}, e^{4\pi i/5}, e^{-4\pi i/5}}$

b)

The roots satisfy $z^5 + 0z^4 + 0z^3 + 0z^2 + 0z - 1 = 0$

Sum of roots $= -\frac{b}{a}$, where b is the coefficient of z^4

Sum of roots $= -\frac{0}{1} = 0$

$\boxed{\text{The five roots add up to 0}}$

c)

Pair each root with its conjugate

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$

$\boxed{\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}}$

d)

Let $c = \cos\frac{2\pi}{5}$, so $\cos\frac{4\pi}{5} = 2c^2 - 1$

$c + 2c^2 - 1 = -\frac{1}{2}$

$4c^2 + 2c - 1 = 0$

$c = \frac{(-2 \pm \sqrt{(4+16)})}{8}$

$c = \frac{(-1 \pm \sqrt{5})}{4}$

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$\frac{2\pi}{5}$ is acute, so $c > 0$

$$\cos\frac{2\pi}{5} = \frac{(\sqrt{5}-1)}{4}$$

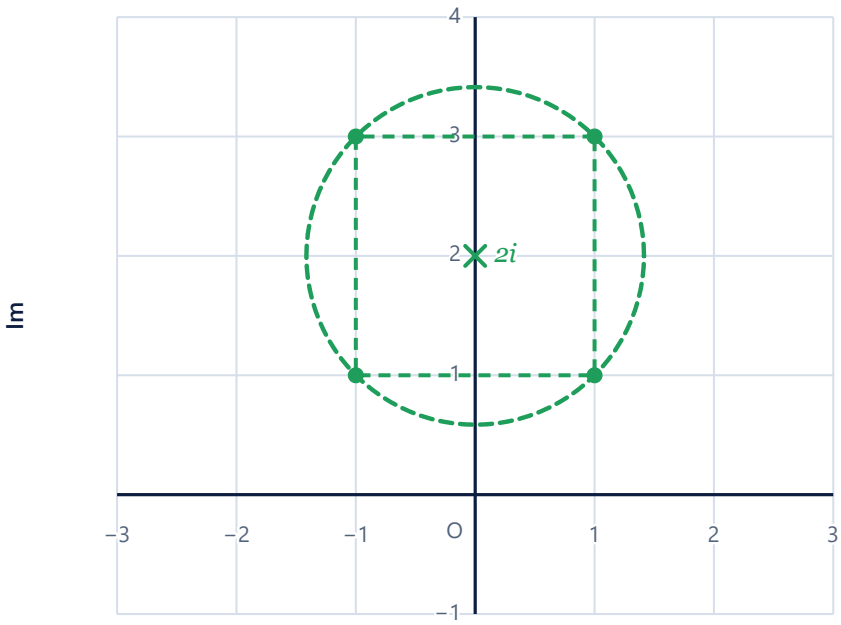
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10

a. Solve the equation $w^4 = -4$, giving your answers in the form $a + bi$, where a and b are real.

b. Hence solve the equation $(z - 2i)^4 = -4$

c. Show the roots from part (b) on an Argand diagram, show that they are the vertices of a square and find the area of the square.



(8 marks)

a)

$-4 = 4e^{\pi i}$

$w^4 = 4e^{i(\pi + 2k\pi)}$

The fourth root of 4 is $\sqrt{2}$

$$w = \sqrt{2}e^{i(\pi/4 + k\pi/2)},\ k = 0, 1, -1, -2$$

$$w = \sqrt{2}e^{\pi i/4}, \sqrt{2}e^{3\pi i/4}, \sqrt{2}e^{-\pi i/4}, \sqrt{2}e^{-3\pi i/4}$$

$$\sqrt{2}e^{\pi i/4} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1 + i$$

The others follow the same way

$$\boxed{w = 1 + i, -1 + i, -1 - i, 1 - i}$$

b)

Let $w = z - 2i$, so $z = w + 2i$

$$\boxed{z = 1 + 3i, -1 + 3i, -1 + i, 1 + i}$$

c)

The roots of $w^4 = -4$ are equally spaced on $|w| = \sqrt{2}$

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Adding $2i$ translates them, so they are equally spaced on $|z - 2i| = \sqrt{2}$

Four points equally spaced on a circle form a square

Side $= 1 - (-1) = 2$

Area $= 2^2$

$$\boxed{\text{Area} = 4}$$

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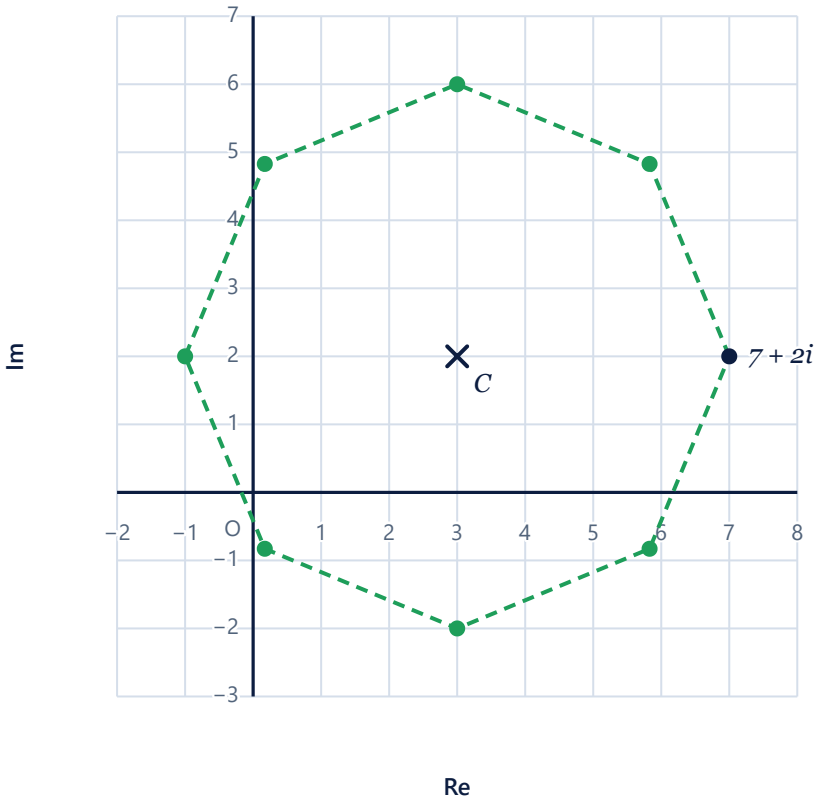
11 An engineer designs a decorative steel panel. On an Argand diagram, where 1 unit represents 10 cm, eight bolts are placed at the vertices of a regular octagon whose centre C represents the complex number $3 + 2i$. One of the bolts is at the point representing $7 + 2i$.

a. Explain why the eight bolts are at the points representing the roots of the equation $(z - 3 - 2i)^8 = 65\,536$

b. Find the complex numbers representing the other seven bolts, giving your answers in exact form $a + bi$.

A strip of steel is fixed in straight lines between adjacent bolts, all the way round the octagon.

c. Find the total length of the strip, giving your answer in cm to 3 significant figures.



(9 marks)

a)

Let $w = z - (3 + 2i)$, the position of a bolt measured from C

$w = (7 + 2i) - (3 + 2i) = 4$ for the given bolt

The bolts are all 4 units from C, equally spaced $\frac{2\pi}{8} = \frac{\pi}{4}$ apart

So the values of w are $4e^{k\pi i/4}$, $k = 0, 1, \ldots, 7$

$(4e^{k\pi i/4})^8 = 4^8 e^{2k\pi i} = 4^8$

$4^8 = 65536$

The bolts satisfy $(z - 3 - 2i)^8 = 65536$

b)

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$$z = 3 + 2i + 4e^{k\pi i/4},\ k = 1, 2, \ldots, 7$$

$$4e^{\pi i/4} = 2\sqrt{2} + 2\sqrt{2}i$$

$$4e^{\pi i/2} = 4i$$

$$4e^{3\pi i/4} = -2\sqrt{2} + 2\sqrt{2}i$$

$$4e^{\pi i} = -4$$

$$4e^{5\pi i/4} = -2\sqrt{2} - 2\sqrt{2}i$$

$$4e^{3\pi i/2} = -4i$$

$$4e^{7\pi i/4} = 2\sqrt{2} - 2\sqrt{2}i$$

Add $3 + 2i$ to each

$$\boxed{(3 + 2\sqrt{2}) + (2 + 2\sqrt{2})i,\ 3 + 6i,\ (3 - 2\sqrt{2}) + (2 + 2\sqrt{2})i,\ -1 + 2i,\ (3 - 2\sqrt{2}) + (2 - 2\sqrt{2})i,\ 3 - 2i,\ (3 + 2\sqrt{2}) + (2 - 2\sqrt{2})i}$$

c)

Each edge faces an angle $\frac{\pi}{4}$ at C, in a triangle with sides 4 and 4

$$\text{edge}^2 = 4^2 + 4^2 - 2 \times 4 \times 4 \times \cos\frac{\pi}{4}$$

$$\text{edge}^2 = 32 - 16\sqrt{2}$$

$$\text{edge} = 3.0614\ldots \text{ units}$$

$$\text{perimeter} = 8 \times 3.0614\ldots = 24.491\ldots \text{ units}$$

1 unit = 10 cm

$$\boxed{\text{Length of strip} = 245 \text{ cm (3 s.f.)}}$$

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12 This question is about the equation $(z + 1)^5 = z^5$

a. Explain why the equation has exactly four roots.

b. Show that each root can be written as $z = \frac{1}{w - 1}$, where w is a fifth root of unity and $w \neq 1$

c. Show that, for $0 < \theta < 2\pi$, $\frac{1}{e^{i\theta} - 1} = -\frac{1}{2} - \frac{1}{2}i \cot \frac{\theta}{2}$

d. Hence write down the four roots of the equation, and state the equation of the line in the Argand diagram on which they all lie.

(10 marks)

a)

$(z+1)^5 = z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1$

The z^5 terms cancel

$5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$

A quartic equation, so exactly four roots

b)

$z = 0$ is not a root, so divide both sides by z^5

$\left(\frac{(z+1)}{z}\right)^5 = 1$

Let $w = \frac{(z+1)}{z}$, a fifth root of unity

$wz = z + 1$

$z(w - 1) = 1$

$w = 1$ would give $0 = 1$, so $w \neq 1$

$z = \frac{1}{(w-1)}$, where $w = e^{2k\pi i/5}$, $k = 1, 2, 3, 4$

c)

Take out a factor of $e^{i\theta/2}$

$e^{i\theta} - 1 = e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})$

$e^{i\phi} - e^{-i\phi} = 2i \sin\phi$

$e^{i\theta} - 1 = 2i \sin\left(\frac{\theta}{2}\right) e^{i\theta/2}$

$\frac{1}{(e^{i\theta} - 1)} = \frac{e^{-i\theta}}{2}/(2i \sin(\theta/2))$

$\frac{1}{i} = -i$, and $e^{-i\theta/2} = \cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right)$

$\frac{1}{(e^{i\theta} - 1)} = \frac{-i(\cos(\theta/2) - i \sin(\theta/2))}{(2\sin(\theta/2))}$

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$$\frac{1}{(e^{i\theta}-1)} = \frac{(-\frac{1}{2} - \sin(\theta/2))/(2\sin(\theta/2))}{}$$

$$\boxed{\frac{1}{(e^{i\theta}-1)} = -\frac{1}{2} - \frac{1}{2}i\cot\left(\frac{\theta}{2}\right)}$$

d)

$\omega = e^{2k\pi i/5}$, so $\theta = \frac{2k\pi}{5}$ and $\frac{\theta}{2} = \frac{k\pi}{5}$

$$\boxed{z = -\frac{1}{2} - \frac{1}{2}i\cot\left(\frac{k\pi}{5}\right),\ k = 1, 2, 3, 4}$$

Every root has real part $-\frac{1}{2}$

$$\boxed{\text{The roots lie on the line } \mathrm{Re}(z) = -\frac{1}{2}}$$

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