Polar coordinates and curves

Worked solutions

Further Maths topic practice

This topic online

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1

a. The point A has polar coordinates $(4, \frac{2\pi}{3})$. Find the exact cartesian coordinates of A.

b. Find, in the form (r, θ) with $r > 0$ and $0 \leq \theta < 2\pi$, the exact polar coordinates of the points with cartesian coordinates

(i) $B(-5, 5)$ (ii) $C(2\sqrt{3}, -2)$

(4 marks)

a)

$x = r\cos\theta,\ y = r\sin\theta$

$$x = 4\cos\frac{2\pi}{3} = 4 \times \left(-\frac{1}{2}\right) = -2$$

$$y = 4\sin\frac{2\pi}{3} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$\boxed{A(-2, 2\sqrt{3})}$$

b)

(i) $r = \sqrt{(x^2 + y^2)}$

$$r = \sqrt{(25 + 25)} = 5\sqrt{2}$$

B is in the second quadrant

$$\tan\alpha = \frac{5}{5} = 1$$

$$\alpha = \frac{\pi}{4}$$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\boxed{B\left(5\sqrt{2}, \frac{3\pi}{4}\right)}$$

(ii)

$$r = \sqrt{(12 + 4)} = 4$$

C is in the fourth quadrant

$$\tan\alpha = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

$$\boxed{C\left(4, \frac{11\pi}{6}\right)}$$

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2 Find a polar equation for each of the following curves, giving your answers in the form $r = f(\theta)$ or $r^2 = f(\theta)$.

a. $x^2 + y^2 = 6x$

b. $x = 4$

c. $xy = 4$

(5 marks)

$x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2$

a)

$r^2 = 6r\cos\theta$

$\boxed{r = 6\cos\theta}$

b)

$r\cos\theta = 4$

$\boxed{r = 4\sec\theta}$

c)

$r\cos\theta \times r\sin\theta = 4$

$r^2\sin\theta\cos\theta = 4$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\frac{1}{2}r^2\sin 2\theta = 4$

$r^2 = \frac{8}{\sin 2\theta}$

$\boxed{r^2 = 8\operatorname{cosec} 2\theta}$

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3 Find a cartesian equation for each of the following curves and describe each curve geometrically.

a. $r = 4 \sin \theta, \quad 0 \leq \theta \leq \pi$

b. $r = \dfrac{2}{\cos \theta + \sin \theta}, \quad -\dfrac{\pi}{4} < \theta < \dfrac{3\pi}{4}$

(5 marks)

a)

Multiply by r

$$r^2 = 4r \sin \theta$$

$$x^2 + y^2 = 4y$$

Complete the square

$$x^2 + (y-2)^2 - 4 = 0$$

$$x^2 + (y-2)^2 = 4$$

Circle, centre (0, 2), radius 2

b)

$$r(\cos \theta + \sin \theta) = 2$$

$$r\cos \theta + r \sin \theta = 2$$

$$x + y = 2$$

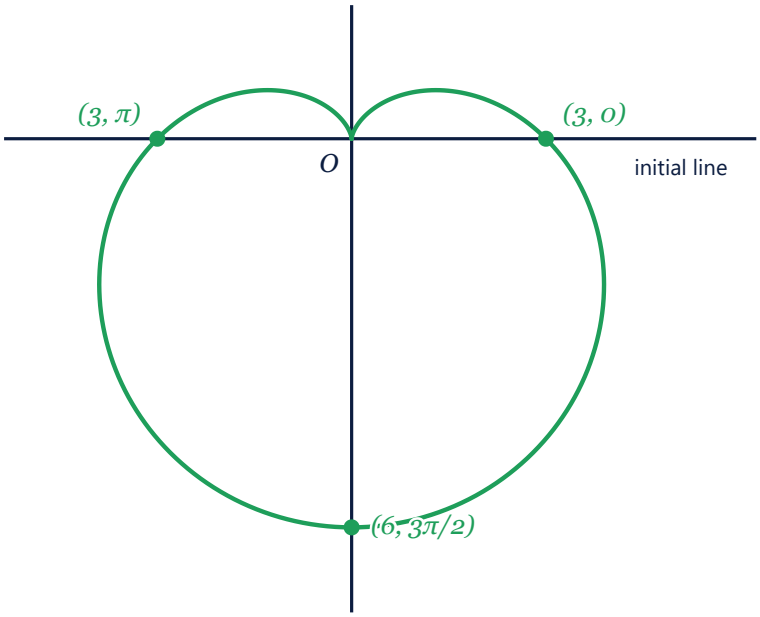
Straight line $x + y = 2$ (gradient -1, through (2, 0) and (0, 2))

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4 Sketch the curve with polar equation

$$r = 3(1 - \sin\theta), \quad 0 \leq \theta < 2\pi$$

On your sketch, label the polar coordinates of the points where the curve meets the initial line, the half-line $\theta = \pi$ and the half-line $\theta = \frac{3\pi}{2}$



(3 marks)

Find r at the key angles

$\theta = 0: r = 3(1 - 0) = 3$

$\theta = \frac{\pi}{2}: r = 3(1 - 1) = 0$

$\theta = \pi: r = 3(1 - 0) = 3$

$\theta = \frac{3\pi}{2}: r = 3(1 + 1) = 6$

$r = 0$ at $\theta = \frac{\pi}{2}$, so the cusp is at the pole

$\sin(\pi - \theta) = \sin\theta$, so symmetrical about $\theta = \frac{\pi}{2}$

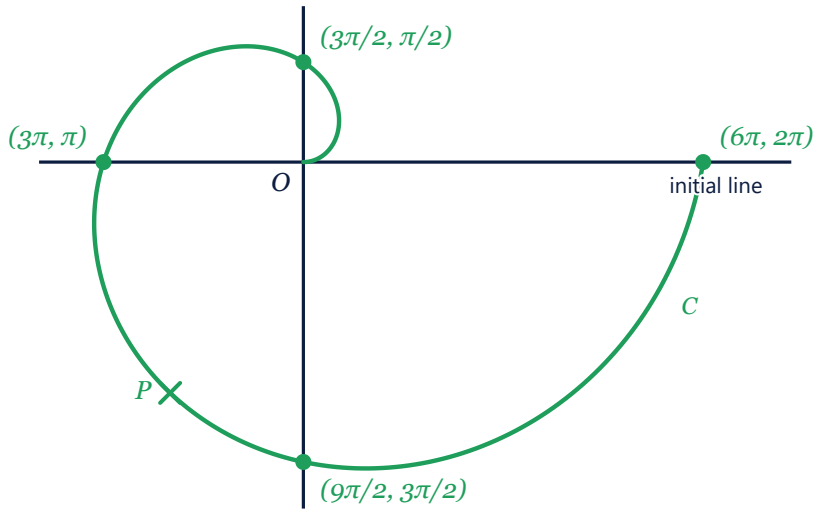
Cardioid through $(3, 0)$, $(3, \pi)$, $(6, \frac{3\pi}{2})$ with its cusp at O

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5 The curve C has polar equation $r = 3\theta$, $0 \leq \theta \leq 2\pi$

a. Sketch C, labelling the polar coordinates of the points where C meets the half-lines $\theta = \frac{\pi}{2}$, $\theta = \pi$, $\theta = \frac{3\pi}{2}$ and the initial line.

b. The point P lies on C where $\theta = \frac{4\pi}{3}$. Find the exact cartesian coordinates of P.



(5 marks)

a)

$\theta = 0$: $r = 0$, start at the pole

$\theta = \frac{\pi}{2}$: $r = \frac{3\pi}{2}$

$\theta = \pi$: $r = 3\pi$

$\theta = \frac{3\pi}{2}$: $r = \frac{9\pi}{2}$

$\theta = 2\pi$: $r = 6\pi$

Spiral from O, anticlockwise, ending at $(6\pi, 2\pi)$ on the initial line

b)

$r = 3 \times \frac{4\pi}{3} = 4\pi$

$x = 4\pi \cos\frac{4\pi}{3} = 4\pi \times (-\frac{1}{2}) = -2\pi$

$y = 4\pi \sin\frac{4\pi}{3} = 4\pi \times (-\frac{\sqrt{3}}{2}) = -2\sqrt{3}\pi$

$P(-2\pi, -2\sqrt{3}\pi)$

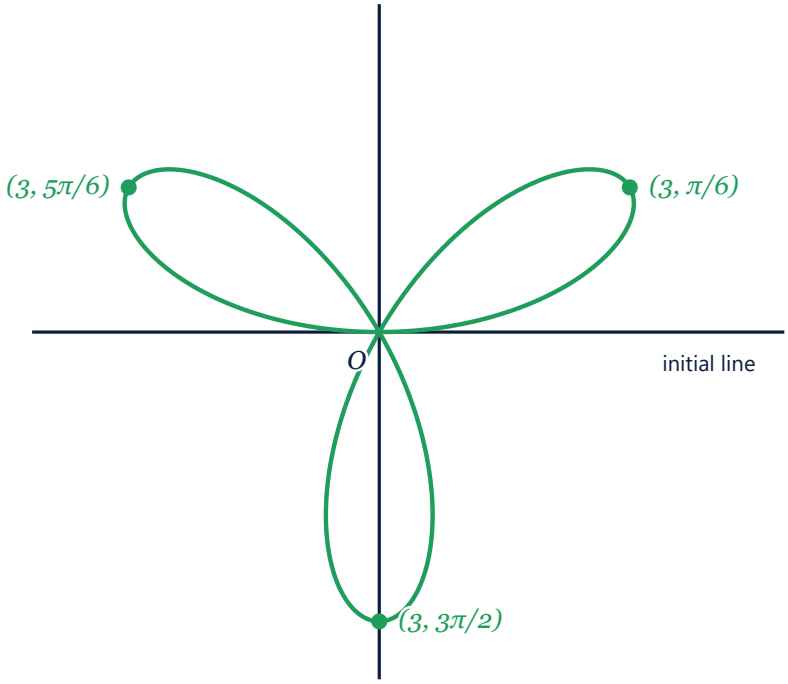
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6 The curve C has polar equation $r = 3 \sin 3\theta$, where $r \geq 0$ and $0 \leq \theta < 2\pi$

a. Explain why there are no points on C for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$

b. Write down the three intervals of θ for which C is defined.

c. Sketch C, showing the polar coordinates of the point of each loop that is furthest from the pole.



(6 marks)

a)

$\frac{\pi}{3} < \theta < \frac{2\pi}{3}$

$\pi < 3\theta < 2\pi$

so $\sin 3\theta < 0$

$r < 0$ there, which is not allowed

b)

Need $\sin 3\theta \geqslant 0$ with $0 \leqslant 3\theta < 6\pi$

$0 \leqslant 3\theta \leqslant \pi,\ 2\pi \leqslant 3\theta \leqslant 3\pi,\ 4\pi \leqslant 3\theta \leqslant 5\pi$

$0 \leqslant \theta \leqslant \frac{\pi}{3},\ \frac{2\pi}{3} \leqslant \theta \leqslant \pi,\ \frac{4\pi}{3} \leqslant \theta \leqslant \frac{5\pi}{3}$

c)

Greatest $r = 3$ when $\sin 3\theta = 1$

$3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

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$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

Three loops from O with tips $(3, \frac{\pi}{6}), (3, \frac{5\pi}{6}), (3, \frac{3\pi}{2})$

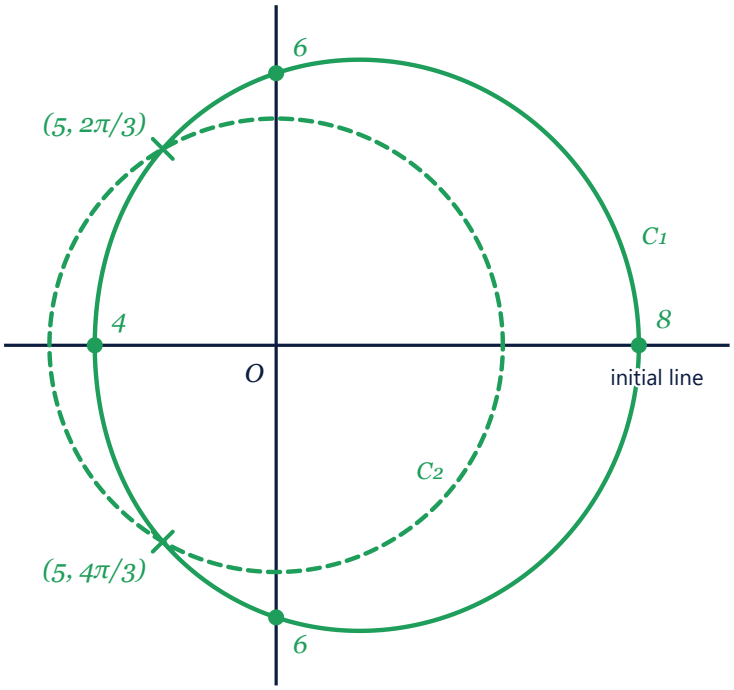
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7 The curves C_1 and C_2 have polar equations

$$C_1: r = 2(3 + \cos\theta), \quad 0 \le \theta < 2\pi \qquad C_2: r = 5, \quad 0 \le \theta < 2\pi$$

a. Sketch C_1 and C_2 on the same diagram, labelling the values of r where C_1 meets the initial line and the half-lines $\theta = \frac{\pi}{2}$, $\theta = \pi$ and $\theta = \frac{3\pi}{2}$

b. Find the exact polar coordinates of the points where C_1 and C_2 meet.



(6 marks)

a)

$\theta = 0: r = 2(3 + 1) = 8$

$\theta = \frac{\pi}{2}: r = 2(3 + 0) = 6$

$\theta = \pi: r = 2(3 - 1) = 4$

$\theta = \frac{3\pi}{2}: r = 6$

$r = a(p + q\cos\theta)$ with $p \geq 2q$ is convex (no dimple)

C2 is a circle, centre O, radius 5

Egg shape through 8, 6, 4, 6 and the circle $r = 5$

b)

$2(3 + \cos\theta) = 5$

$\cos\theta = -\frac{1}{2}$

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$$\theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$

$$\left(5, \frac{2\pi}{3}\right) \text{ and } \left(5, \frac{4\pi}{3}\right)$$

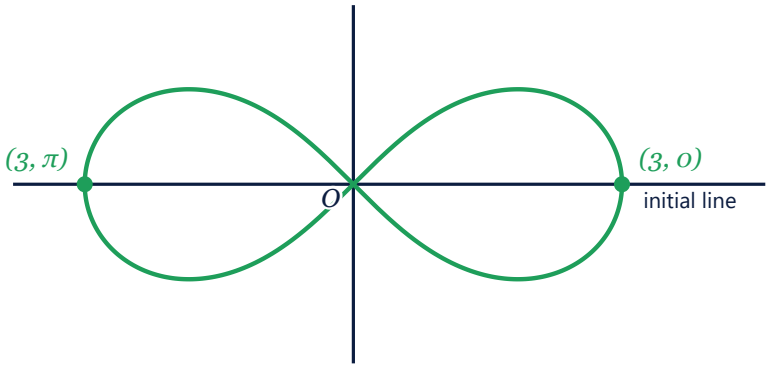
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8 The curve C has polar equation $r^2 = 9\cos 2\theta$, $0 \le \theta < 2\pi$

a. State the set of values of θ for which C exists.

b. Sketch C, labelling the points where it meets the initial line and the half-line $\theta = \pi$

c. Show that a cartesian equation of C is $(x^2 + y^2)^2 = 9(x^2 - y^2)$



(6 marks)

a)

$r^2 \geqslant 0$ so $\cos 2\theta \geqslant 0$

$$0 \leqslant 2\theta \leqslant \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \leqslant 2\theta \leqslant \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \leqslant 2\theta < 4\pi$$

$$\boxed{0 \leqslant \theta \leqslant \frac{\pi}{4}, \frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4}, \frac{7\pi}{4} \leqslant \theta < 2\pi}$$

b)

$\theta = 0: r^2 = 9, r = 3$

$\theta = \pi: r^2 = 9, r = 3$

$\theta = \pm\frac{\pi}{4}: r = 0$

$\boxed{\text{Two loops meeting at O, through (3, 0) and (3, } \pi)}$

c)

Multiply by r^2

$r^4 = 9r^2\cos 2\theta$

$r^4 = 9(r^2\cos^2\theta - r^2\sin^2\theta)$

$r^2 = x^2 + y^2$, $r\cos\theta = x$, $r\sin\theta = y$

$\boxed{(x^2 + y^2)^2 = 9(x^2 - y^2) \text{ (shown)}}$

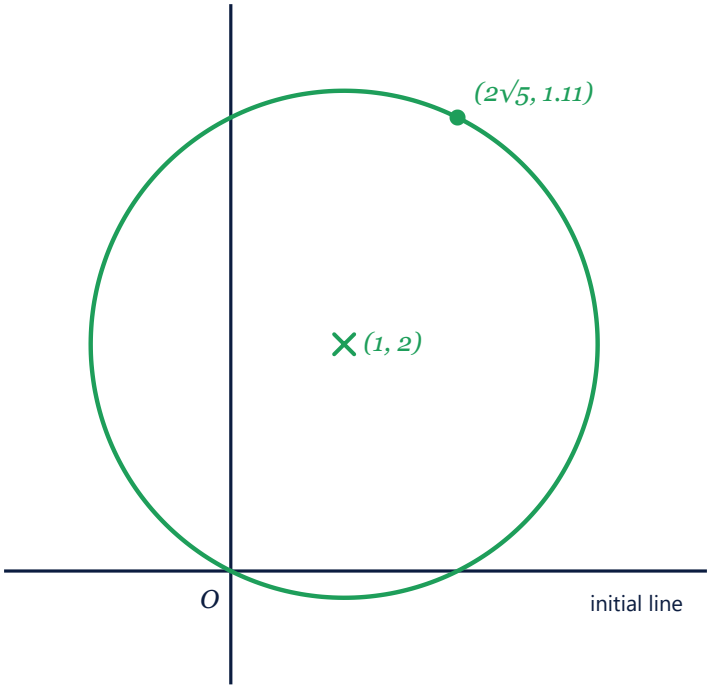
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9 The curve C has polar equation $r = 2\cos\theta + 4\sin\theta$, for values of θ where $r \geq 0$

a. Show that C is a circle, and find its centre and its radius.

b. Sketch C.

c. By writing $2\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$, find the greatest value of r and the value of θ, to 3 significant figures, at which it occurs.



(8 marks)

a)

Multiply by r

$$r^2 = 2r\cos\theta + 4r\sin\theta$$

$$x^2 + y^2 = 2x + 4y$$

Complete the square

$$(x-1)^2 - 1 + (y-2)^2 - 4 = 0$$

$$(x-1)^2 + (y-2)^2 = 5$$

Circle, centre (1, 2), radius $\sqrt{5}$

b)

(0, 0) satisfies the equation, so the circle passes through the pole

Circle through O with centre (1, 2)

c)

$$R\cos\alpha = 2$$

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$$R\sin\alpha = 4$$

$$R = \sqrt{(4+16)} = 2\sqrt{5}$$

$$\tan\alpha = \frac{4}{2} = 2$$

$$\alpha = 1.107...$$

$$r = 2\sqrt{5}\cos(\theta - 1.107...)$$

Greatest when $\cos(\theta - \alpha) = 1$

$$r = 2\sqrt{5} \text{ when } \theta = 1.11$$

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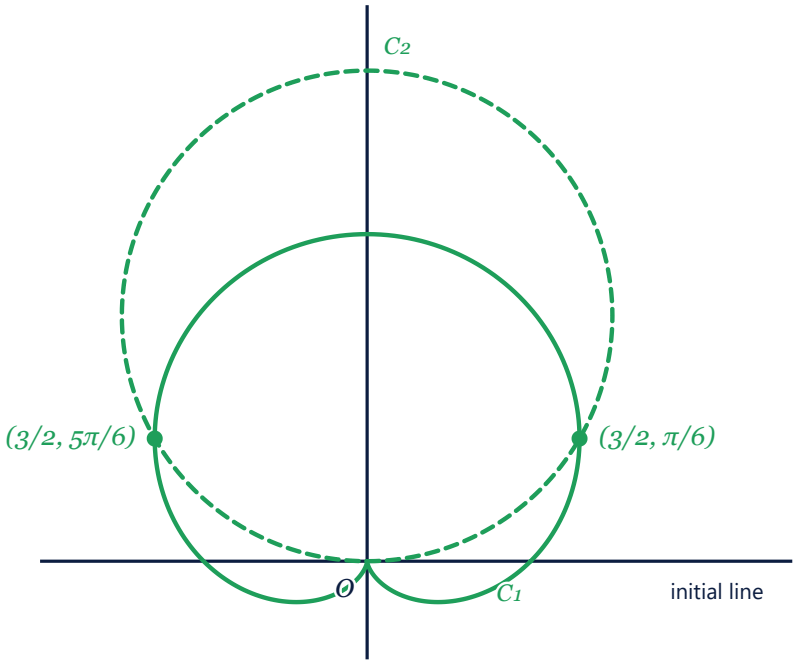
10 The curves C_1 and C_2 have polar equations

$$C_1: r = 1 + \sin\theta, \quad 0 \le \theta < 2\pi \qquad C_2: r = 3\sin\theta, \quad 0 \le \theta \le \pi$$

a. Sketch C_1 and C_2 on the same diagram.

b. Find the polar coordinates of the two points, other than the pole, where the curves meet.

c. Show that these two points lie on the line with cartesian equation $y = \frac{3}{4}$, and find the exact distance between them.



(8 marks)

a)

$C_1: r = 1 + \sin\theta$

$\theta = 0: r = 1$

$\theta = \frac{\pi}{2}: r = 2$

$\theta = \pi: r = 1$

$\theta = \frac{3\pi}{2}: r = 0$

$C_2: r^2 = 3r\sin\theta$ gives $x^2 + (y - \frac{3}{2})^2 = \frac{9}{4}$

C_1 cardioid with cusp at O; C_2 circle centre $(0, \frac{3}{2})$ radius $\frac{3}{2}$

b)

$1 + \sin\theta = 3\sin\theta$

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$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$r = 3 \times \frac{1}{2} = \frac{3}{2}$$

$$\boxed{\left(\frac{3}{2}, \frac{\pi}{6}\right) \text{ and } \left(\frac{3}{2}, \frac{5\pi}{6}\right)}$$

c)

$$y = r\sin\theta = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4} \text{ at both points}$$

so both lie on $y = \frac{3}{4}$ (shown)

$$x = \frac{3}{2}\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{4}$$

$$x = \frac{3}{2}\cos\frac{5\pi}{6} = -\frac{3\sqrt{3}}{4}$$

$$\text{distance} = \frac{3\sqrt{3}}{4} - \left(-\frac{3\sqrt{3}}{4}\right)$$

$$\boxed{\text{distance} = \frac{3\sqrt{3}}{2}}$$

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11 The edge of a flower bed is modelled by the curve with polar equation

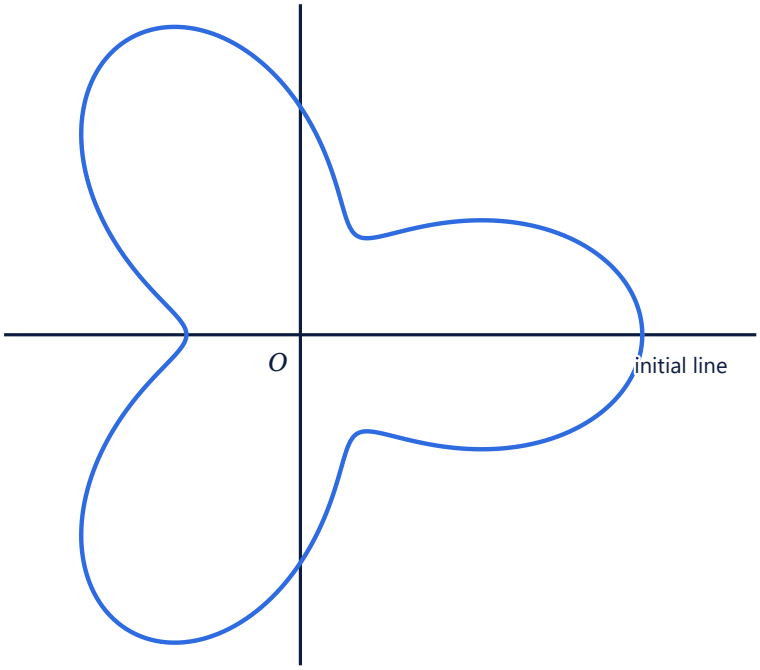
$$r = 2 + \cos 3\theta, \quad 0 \leq \theta < 2\pi$$

where r is measured in metres and the pole O is a fountain in the middle of the bed, as shown in the diagram.

a. Find the greatest and least distances from the fountain to the edge of the bed, and the values of θ at which the greatest distance occurs.

b. A straight path is laid along the initial line, passing through O and crossing the bed from edge to edge. Find the length of path inside the bed.

c. Lights are placed at the three points of the edge that are furthest from O. Find the distance between two of these lights, giving your answer in metres to 3 significant figures.



(7 marks)

a)

$-1 \leq \cos 3\theta \leq 1$

$1 \leq r \leq 3$

Greatest when $\cos 3\theta = 1$

$3\theta = 0, 2\pi, 4\pi$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Greatest 3 m at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$; least 1 m

b)

The path is along $\theta = 0$ and $\theta = \pi$

$\theta = 0$: $r = 2 + 1 = 3$

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$\theta = \pi : r = 2 + \cos 3\pi = 1$

Length = 3 + 1 = 4 m

c)

Lights at $(3, 0)$ and $(3, \frac{2\pi}{3})$ in polar form

$$x = 3\cos\frac{2\pi}{3} = -\frac{3}{2}$$

$$y = 3\sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$$

$$d^2 = (3 + \frac{3}{2})^2 + (\frac{3\sqrt{3}}{2})^2$$

$$d^2 = \frac{81}{4} + \frac{27}{4} = 27$$

$$d = 3\sqrt{3} = 5.196...$$

5.20 m

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12 The line l has polar equation $r \sin(\theta + \frac{\pi}{3}) = 3$

a. Show that a cartesian equation of l is $\sqrt{3}x + y = 6$

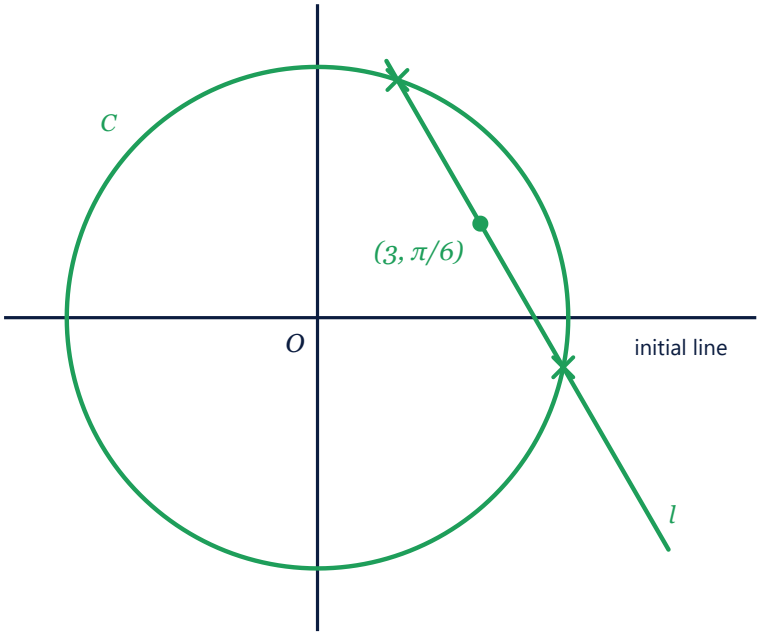
b. Find the polar coordinates of the point of l that is closest to the pole.

The circle C has polar equation $r = 4$

c. Sketch l and C on the same diagram.

d. Find the exact length of the chord that l cuts off from C.

e. Find the polar coordinates of the points where l meets C, giving values of θ in the interval $0 \leq \theta < 2\pi$ to 3 significant figures.



(9 marks)

a)

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$r \sin\theta \cos\frac{\pi}{3} + r \cos\theta \sin\frac{\pi}{3} = 3$$

$$\frac{1}{2}y + \frac{\sqrt{3}}{2}x = 3$$

$$\boxed{\sqrt{3}x + y = 6 \text{ (shown)}}$$

b)

Shortest distance from O to the line

$$d = \frac{6}{\sqrt{(3 + 1)}} = 3$$

r is least when $\sin(\theta + \frac{\pi}{3}) = 1$

$$\theta + \frac{\pi}{3} = \frac{\pi}{2}$$

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$$\theta = \frac{\pi}{6}$$

$\left(3, \frac{\pi}{6}\right)$

c)

l meets the axes at $(2\sqrt{3}, 0)$ and $(0, 6)$; C is the circle radius 4 about O

Line cutting the circle twice

d)

The perpendicular from O bisects the chord

$$\left(\frac{1}{2}\text{chord}\right)^2 = 4^2 - 3^2 = 7$$

$\text{chord} = 2\sqrt{7}$

e)

$$4\sin\left(\theta + \frac{\pi}{3}\right) = 3$$

$$\sin\left(\theta + \frac{\pi}{3}\right) = \frac{3}{4}$$

$$\theta + \frac{\pi}{3} = 0.8481 \text{ or } \pi - 0.8481 = 2.2935$$

$$\theta = -0.1991 \text{ or } \theta = 1.2463$$

Add 2π to the negative value

$$\theta = 6.0841$$

(4, 1.25) and (4, 6.08)

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