Second order homogeneous differential equations

Worked solutions

Further Maths topic practice

This topic online

1 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

(3 marks)

Auxiliary equation

$$m^2 - 5m + 6 = 0$$

$$(m - 2)(m - 3) = 0$$

$$m = 2 \text{ or } m = 3$$

Distinct real roots: $y = Ae^{\alpha x} + Be^{\beta x}$

$$\boxed{y = Ae^{2x} + Be^{3x}}$$

2 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

(3 marks)

Auxiliary equation

$$m^2 + 6m + 9 = 0$$

$$(m + 3)^2 = 0$$

$m = -3$ (repeated)

Repeated root: $y = (A + Bx)e^{\alpha x}$

$$\boxed{y = (A + Bx)e^{-3x}}$$

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3 Find the general solution of each differential equation.

a. $\frac{d^2y}{dx^2} + 16y = 0$

b. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

(4 marks)

a) Auxiliary equation

$m^2 + 16 = 0$

$m = \pm 4i$

Roots $\pm qi$: $y = A\cos qx + B\sin qx$

$$\boxed{y = A\cos 4x + B\sin 4x}$$

b) Auxiliary equation

$m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{(16 - 52)}}{2}$$

$$m = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

Roots $p \pm qi$: $y = e^{px}(A\cos qx + B\sin qx)$

$$\boxed{y = e^{-2x}(A\cos 3x + B\sin 3x)}$$

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4 The differential equation

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

has a particular solution for which $y = 3$ and $\frac{dy}{dx} = 1$ when $x = 0$

a. Find this particular solution.

b. Find the exact value of y when $x = \ln 4$

(6 marks)

a) Auxiliary equation

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = \frac{3}{2} \text{ or } m = -2$$

$$y = Ae^{3x/2} + Be^{-2x}$$

$$\frac{dy}{dx} = \frac{3}{2}Ae^{3x/2} - 2Be^{-2x}$$

Use $y = 3$ and $\frac{dy}{dx} = 1$ when $x = 0$

$$A + B = 3$$

$$\frac{3}{2}A - 2B = 1$$

Substitute $B = 3 - A$

$$\frac{3}{2}A - 6 + 2A = 1$$

$$\frac{7}{2}A = 7$$

$$A = 2, B = 1$$

$$\boxed{y = 2e^{3x/2} + e^{-2x}}$$

b) $x = \ln 4$

$$e^{3x/2} = e^{(3/2)\ln 4} = 4^{3/2} = 8$$

$$e^{-2x} = e^{-2\ln 4} = 4^{-2} = \frac{1}{16}$$

$$y = 2 \times 8 + \frac{1}{16}$$

$$\boxed{y = \frac{257}{16}}$$

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5 The curve C has equation $y = f(x)$, where y satisfies the differential equation

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

The curve passes through the point (0, 2), where the gradient of C is 0

a. Find $f(x)$.

b. Find the coordinates of the point where C crosses the x-axis.

c. Show that (0, 2) is the only stationary point of C and that it is a maximum.

(7 marks)

a) Auxiliary equation

$$4m^2 - 4m + 1 = 0$$

$$(2m - 1)^2 = 0$$

$$m = \frac{1}{2} \text{ (repeated)}$$

$$y = (A + Bx)e^{x/2}$$

$y = 2$ when $x = 0$

$$A = 2$$

Product rule

$$\frac{dy}{dx} = Be^{x/2} + \frac{1}{2}(A + Bx)e^{x/2}$$

$\frac{dy}{dx} = 0$ when $x = 0$

$$0 = B + \frac{1}{2}(2)$$

$$B = -1$$

$$\boxed{f(x) = (2 - x)e^{x/2}}$$

b) $e^{x/2} > 0$ for all x

$$2 - x = 0$$

$$\boxed{(2, 0)}$$

c)

$$\frac{dy}{dx} = -e^{x/2} + \frac{1}{2}(2 - x)e^{x/2}$$

$$\frac{dy}{dx} = -\frac{x}{2}e^{x/2}$$

$\frac{dy}{dx} = 0$ only when $x = 0$, so (0, 2) is the only stationary point

$$\frac{d^2y}{dx^2} = -\frac{1}{2}e^{x/2} - \frac{x}{4}e^{x/2}$$

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$\frac{d^2y}{dx^2} = -\frac{1}{2}$ at $x = 0$

$\frac{d^2y}{dx^2} < 0$, so $(0, 2)$ is a maximum

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6 The differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

has a particular solution for which $y = 1$ and $\frac{dy}{dx} = 3$ when $x = 0$

a. Find this particular solution.

b. Find the smallest positive value of x for which $y = 0$, giving your answer to 3 significant figures. (7 marks)

a) Auxiliary equation

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{(4 - 20)}}{2}$$

$$m = -1 \pm 2i$$

$$y = e^{-x}(A \cos 2x + B \sin 2x)$$

$y = 1$ when $x = 0$

$$A = 1$$

Product rule

$$\frac{dy}{dx} = -e^{-x}(A \cos 2x + B \sin 2x) + e^{-x}(-2A \sin 2x + 2B \cos 2x)$$

$\frac{dy}{dx} = 3$ when $x = 0$

$$3 = -A + 2B$$

$$3 = -1 + 2B$$

$$B = 2$$

$$\boxed{y = e^{-x}(\cos 2x + 2 \sin 2x)}$$

b) $e^{-x} > 0$, so the bracket is zero

$$\cos 2x + 2 \sin 2x = 0$$

$$\tan 2x = -\frac{1}{2}$$

tan is negative in the second quadrant first

$$2x = \pi - \arctan\frac{1}{2}$$

$$2x = 2.6779...$$

$$\boxed{x = 1.34 \text{ (3 s.f.)}}$$

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7 The differential equation

$$\frac{d^2y}{dx^2} + 9y = 0$$

has a particular solution for which $y = 2$ when $x = 0$ and $y = 5$ when $x = \frac{\pi}{6}$

a. Find this particular solution.

b. Write your answer to part (a) in the form $y = R\cos(3x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving the exact value of R and the value of α to 3 significant figures.

c. Hence find the maximum value of y and the smallest positive value of x at which it occurs.

(7 marks)

a) Auxiliary equation

$$m^2 + 9 = 0$$

$$m = \pm 3i$$

$$y = A\cos 3x + B\sin 3x$$

$x = 0, y = 2$

$$A = 2$$

$x = \frac{\pi}{6}, y = 5$: $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

$$5 = 2(0) + B(1)$$

$$B = 5$$

$$\boxed{y = 2\cos 3x + 5\sin 3x}$$

b) Compare with $R\cos\alpha\cos 3x + R\sin\alpha\sin 3x$

$$R\cos\alpha = 2$$

$$R\sin\alpha = 5$$

$$R = \sqrt{(2^2 + 5^2)} = \sqrt{29}$$

$$\tan\alpha = \frac{5}{2}$$

$$\alpha = 1.1902...$$

$$\boxed{y = \sqrt{29}\cos(3x - 1.19)}$$

c) Maximum when $\cos(3x - \alpha) = 1$

$$3x - \alpha = 0$$

$$x = \frac{1.1902...}{3}$$

$$\boxed{\text{Maximum } y = \sqrt{29} \text{ when } x = 0.397 \text{ (3 s.f.)}}$$

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8 The differential equation

$$\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + 9y = 0$$

where k is a positive constant, is to be solved.

a. Find the value of k for which the auxiliary equation has a repeated root, and write down the general solution of the differential equation for this value of k.

b. Find the general solution of the differential equation when $k = 5$

c. Find the general solution of the differential equation when $k = 2$

d. State the set of values of k for which the general solution involves sines and cosines.

(8 marks)

Auxiliary equation

$$m^2 + 2km + 9 = 0$$

$$\text{discriminant} = 4k^2 - 36$$

a) Repeated root: discriminant = 0

$$4k^2 = 36$$

$$k = 3 \ (k > 0)$$

$$m^2 + 6m + 9 = (m + 3)^2 = 0$$

$$\boxed{k = 3,\ y = (A + Bx)e^{-3x}}$$

b) $k = 5$

$$m^2 + 10m + 9 = 0$$

$$(m + 1)(m + 9) = 0$$

$$m = -1 \text{ or } m = -9$$

$$\boxed{y = Ae^{-x} + Be^{-9x}}$$

c) $k = 2$

$$m^2 + 4m + 9 = 0$$

$$m = \frac{-4 \pm \sqrt{(16 - 36)}}{2}$$

$$m = -2 \pm \sqrt{5}\,i$$

$$\boxed{y = e^{-2x}(A\cos\sqrt{5}x + B\sin\sqrt{5}x)}$$

d) Complex roots: discriminant < 0

$$4k^2 - 36 < 0$$

$$k^2 < 9$$

$$\boxed{0 < k < 3}$$

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9

a. The general solution of a differential equation of the form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

is $y = Ae^{-\frac{1}{2}x} + Be^{3x}$

Find possible integer values of a, b and c.

b. Find a differential equation of the form $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$ whose general solution is

$y = e^{-x}(A\cos 3x + B\sin 3x)$

c. Show, by substituting directly into the differential equation, that $y = xe^{4x}$ is a solution of

$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

(7 marks)

a) Roots of the auxiliary equation are $-\frac{1}{2}$ and 3

$(2m + 1)(m - 3) = 0$

$2m^2 - 5m - 3 = 0$

$a = 2, b = -5, c = -3$

b) Roots $-1 \pm 3i$

sum of roots $= -2$

product of roots $= (-1)^2 + 3^2 = 10$

$m^2 + 2m + 10 = 0$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$

c) Product rule

$y = xe^{4x}$

$\frac{dy}{dx} = e^{4x} + 4xe^{4x}$

$\frac{d^2y}{dx^2} = 4e^{4x} + 4e^{4x} + 16xe^{4x} = 8e^{4x} + 16xe^{4x}$

Substitute

$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8e^{4x} + 16xe^{4x} - 8e^{4x} - 32xe^{4x} + 16xe^{4x}$

$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

So $y = xe^{4x}$ is a solution (shown)

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10 A metal rod of length 8 cm sticks out from a hot wall. At a point x cm from the wall, the temperature of the rod is θ °C above the temperature of the surrounding air. The temperature is modelled by the differential equation

$$\frac{d^2\theta}{dx^2} = 0.25\theta, \quad 0 \le x \le 8$$

At the wall, θ = 60. The free end of the rod is insulated, so that $\frac{d\theta}{dx} = 0$ when $x = 8$

a. Find the general solution of the differential equation.

b. Show that the particular solution can be written as

$$\theta = \frac{60\cosh(4 - \frac{1}{2}x)}{\cosh 4}$$

c. Find, according to the model, the temperature of the free end of the rod above the temperature of the air.

d. Find the distance from the wall of the point on the rod where the temperature is 10 °C above the temperature of the air.

(10 marks)

a) Auxiliary equation

$$m^2 - 0.25 = 0$$

$$m = \pm\frac{1}{2}$$

$$\theta = Ae^{x/2} + Be^{-x/2}$$

b) Differentiate

$$\frac{d\theta}{dx} = \frac{1}{2}Ae^{x/2} - \frac{1}{2}Be^{-x/2}$$

$\frac{d\theta}{dx} = 0$ when $x = 8$

$$Ae^4 - Be^{-4} = 0$$

$$B = Ae^8$$

$\theta = 60$ when $x = 0$

$$A + Ae^8 = 60$$

$$A = \frac{60}{(1 + e^8)}$$

$$\theta = A(e^{x/2} + e^{8 - x/2})$$

$$\theta = Ae^4(e^{x/2 - 4} + e^{4 - x/2})$$

$$\theta = 2Ae^4\cosh(4 - \frac{x}{2})$$

$$2Ae^4 = \frac{120e^4}{(1 + e^8)} = \frac{120}{(e^{-4} + e^4)} = \frac{60}{\cosh 4}$$

$$\theta = \frac{60\cosh(4 - x}{2)/\cosh 4} \text{ (shown)}$$

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c) $x = 8$, $\cosh 0 = 1$

$$\theta = \frac{60}{\cosh 4}$$

$$\boxed{\theta = 2.20\ ^\circ C\ (3\ s.f.)}$$

d) $\theta = 10$

$$\cosh\left(4 - \frac{x}{2}\right) = \frac{\cosh 4}{6}$$

$$\cosh\left(4 - \frac{x}{2}\right) = 4.5514...$$

$0 \leqslant x \leqslant 8$, so $4 - \frac{x}{2} \geqslant 0$: take the positive arcosh

$$4 - \frac{x}{2} = \operatorname{arcosh} 4.5514...$$

$$4 - \frac{x}{2} = 2.1963...$$

$$\boxed{x = 3.61\ cm\ (3\ s.f.)}$$

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11 The differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

has a particular solution for which $y = 0$ and $\frac{dy}{dx} = 4$ when $x = 0$

a. Show that this particular solution is $y = 4e^{-x} \sin x$

b. Show that the stationary points of the curve $y = 4e^{-x} \sin x$, $x \geq 0$, occur when $x = \frac{\pi}{4} + n\pi$, where n is a non-negative integer.

c. Find the exact value of y at the first maximum point, for which $x = \frac{\pi}{4}$

d. The next maximum point is at $x = \frac{9\pi}{4}$. Show that the ratio of the value of y at the first maximum to the value of y at this maximum is $e^{2\pi}$

(10 marks)

a) Auxiliary equation

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{(4 - 8)}}{2} = -1 \pm i$$

$$y = e^{-x}(A \cos x + B \sin x)$$

$y = 0$ when $x = 0$

$$A = 0$$

$$y = Be^{-x} \sin x$$

$$\frac{dy}{dx} = -Be^{-x} \sin x + Be^{-x} \cos x$$

$\frac{dy}{dx} = 4$ when $x = 0$

$$B = 4$$

$$\boxed{y = 4e^{-x} \sin x \text{ (shown)}}$$

b) Stationary points: $\frac{dy}{dx} = 0$

$$4e^{-x}(\cos x - \sin x) = 0$$

$e^{-x} \neq 0$

$$\sin x = \cos x$$

$$\tan x = 1$$

$$\boxed{x = \frac{\pi}{4} + n\pi \text{ (shown)}}$$

c) $x = \frac{\pi}{4}$

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$$y = 4e^{-\pi/4} \times \frac{\sqrt{2}}{2}$$

$$\boxed{y = 2\sqrt{2}\, e^{-\pi/4}}$$

d) $x = \frac{9\pi}{4}$, $\sin\frac{9\pi}{4} = \frac{\sqrt{2}}{2}$

$$y = 2\sqrt{2}\, e^{-9\pi/4}\}$$

$$\text{ratio} = \frac{2\sqrt{2}\, e^{-\pi/4}}{2\sqrt{2}\, e^{-9\pi/4}}\}$$

$$\text{ratio} = e^{-\pi/4 + 9\pi/4}$$

$$\boxed{\text{ratio} = e^{2\pi} \text{ (shown)}}$$

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12 A straight smooth wire of length 3 m rotates in a horizontal plane about one end, O. A small bead is threaded on the wire. At time t seconds the bead is r metres from O, where

$$\frac{d^2r}{dt^2} = 4r$$

Initially the bead is 0.5 m from O and at rest relative to the wire.

a. Show that $r = 0.5 \cosh 2t$

b. Find the time taken for the bead to reach the end of the wire.

c. Find the exact speed of the bead along the wire as it reaches the end of the wire.

In a second experiment the bead again starts 0.5 m from O, but is projected towards O with speed 1 m s^{-1} along the wire.

d. Find r in terms of t for the second experiment.

e. Explain what the model predicts for the motion of the bead in the second experiment.

(11 marks)

a) Auxiliary equation

$$m^2 - 4 = 0$$

$$m = \pm 2$$

$$r = Ae^{2t} + Be^{-2t}$$

$$\frac{dr}{dt} = 2Ae^{2t} - 2Be^{-2t}$$

$t = 0$: $r = 0.5$, $\frac{dr}{dt} = 0$

$$A + B = 0.5$$

$$2A - 2B = 0$$

$$A = B = 0.25$$

$$r = 0.25(e^{2t} + e^{-2t})$$

$\boxed{r = 0.5 \cosh 2t \text{ (shown)}}$

b) $r = 3$

$$0.5 \cosh 2t = 3$$

$$\cosh 2t = 6$$

$$t = \frac{1}{2}\operatorname{arcosh} 6 = \frac{1}{2}\ln(6 + \sqrt{35})$$

$\boxed{t = 1.24 \text{ s (3 s.f.)}}$

c) Differentiate

$$\frac{dr}{dt} = \sinh 2t$$

Use $\cosh^2 - \sinh^2 = 1$

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$\sinh 2t = \sqrt{(6^2 - 1)}$

$\boxed{\text{speed} = \sqrt{35}\ \text{m s}^{-1}}$

d) $t = 0$: $r = 0.5$, $\frac{dr}{dt} = -1$

$$A + B = 0.5$$
$$2A - 2B = -1$$
$$A - B = -0.5$$
$$A = 0, B = 0.5$$

$\boxed{r = 0.5e^{-2t}}$

e)

r decreases for all t and $r \to 0$ as $t \to \infty$, but $r > 0$

The bead moves towards O, getting ever closer, but never reaches it

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