Second order non-homogeneous differential equations

Worked solutions

Further Maths topic practice

This topic online

1 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x$$

(4 marks)

Complementary function: auxiliary equation

$$m^2 - 3m + 2 = 0$$

$$(m - 1)(m - 2) = 0$$

$$CF: y = Ae^x + Be^{2x}$$

Particular integral: try $y = ax + b$

$$\frac{dy}{dx} = a, \quad \frac{d^2y}{dx^2} = 0$$

$$0 - 3a + 2(ax + b) = 4x$$

Compare coefficients

$$x: 2a = 4, \text{ so } a = 2$$

$$\text{constant: } 2b - 3a = 0, \text{ so } b = 3$$

$$PI: y = 2x + 3$$

General solution = CF + PI

$$\boxed{y = Ae^x + Be^{2x} + 2x + 3}$$

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2 Find the general solution of the differential equation

$\frac{d^2y}{dx^2} + 4y = 15 \sin x$

(4 marks)

CF: auxiliary equation

$m^2 + 4 = 0$

$m = \pm 2i$

CF: $y = A\cos 2x + B\sin 2x$

PI: try $y = \lambda \sin x + \mu \cos x$

$\frac{d^2y}{dx^2} = -\lambda \sin x - \mu \cos x$

$-\lambda \sin x - \mu \cos x + 4\lambda \sin x + 4\mu \cos x = 15 \sin x$

$3\lambda \sin x + 3\mu \cos x = 15 \sin x$

$\lambda = 5, \mu = 0$

PI: $y = 5 \sin x$

$$y = A\cos 2x + B\sin 2x + 5\sin x$$

3 Find the general solution of the differential equation

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$

(5 marks)

CF

$m^2 + 2m - 3 = 0$

$(m-1)(m+3) = 0$

CF: $y = Ae^x + Be^{-3x}$

PI: e^{2x} is not in the CF, so try $y = \lambda e^{2x}$

$\frac{dy}{dx} = 2\lambda e^{2x}$

$\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

$4\lambda e^{2x} + 4\lambda e^{2x} - 3\lambda e^{2x} = 10e^{2x}$

$5\lambda = 10$

$\lambda = 2$

$$y = Ae^x + Be^{-3x} + 2e^{2x}$$

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4 The differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{3x}$$

is to be solved.

a. Find the complementary function and explain why a particular integral of the form $y = \lambda e^{3x}$ cannot be used.

b. Using a particular integral of the form $y = \lambda x e^{3x}$, find the general solution of the differential equation.

(6 marks)

a)

$$m^2 - m - 6 = 0$$

$$(m - 3)(m + 2) = 0$$

CF: $y = Ae^{3x} + Be^{-2x}$

λe^{3x} is part of the CF, so it satisfies the equation with 0 on the right

Substituting λe^{3x} gives 0, never $10e^{3x}$

b) Try $y = \lambda x e^{3x}$, product rule

$$\frac{dy}{dx} = \lambda e^{3x} + 3\lambda x e^{3x}$$

$$\frac{d^2y}{dx^2} = 3\lambda e^{3x} + 3\lambda e^{3x} + 9\lambda x e^{3x} = 6\lambda e^{3x} + 9\lambda x e^{3x}$$

Substitute

$$(6\lambda + 9\lambda x) - (\lambda + 3\lambda x) - 6\lambda x = 10 \text{ (dividing by } e^{3x})$$

$$5\lambda = 10$$

$$\lambda = 2$$

$$y = Ae^{3x} + Be^{-2x} + 2xe^{3x}$$

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5 The differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2$$

is to be solved.

a. Find the general solution of the differential equation.

b. Find the particular solution for which $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$

(8 marks)

a) CF

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$\text{CF: } y = (A + Bx)e^{2x}$$

PI: try $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

$$2a - 4(2ax + b) + 4(ax^2 + bx + c) = 8x^2$$

Compare coefficients

$$x^2\text{: } 4a = 8, \text{ so } a = 2$$

$$x\text{: } 4b - 8a = 0, \text{ so } b = 4$$

$$\text{constant: } 2a - 4b + 4c = 0$$

$$4 - 16 + 4c = 0, \text{ so } c = 3$$

$$\boxed{y = (A + Bx)e^{2x} + 2x^2 + 4x + 3}$$

b) $y = 1$ when $x = 0$

$$A + 3 = 1$$

$$A = -2$$

$$\frac{dy}{dx} = Be^{2x} + 2(A + Bx)e^{2x} + 4x + 4$$

$\frac{dy}{dx} = 2$ when $x = 0$

$$B + 2A + 4 = 2$$

$$B - 4 + 4 = 2$$

$$B = 2$$

$$\boxed{y = (2x - 2)e^{2x} + 2x^2 + 4x + 3}$$

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6 The differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10 \sin x$$

is to be solved.

a. Find a particular integral of the differential equation.

b. Find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

c. Describe the behaviour of y for large values of x, stating the amplitude of the oscillation as an exact value.

(9 marks)

a) Try $y = a\cos x + b\sin x$

$$\frac{dy}{dx} = -a\sin x + b\cos x$$

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x$$

Substitute and compare coefficients

$\cos x$: $-a + 2b + 5a = 0$, so $4a + 2b = 0$

$\sin x$: $-b - 2a + 5b = 10$, so $4b - 2a = 10$

$b = -2a$

$-8a - 2a = 10$

$a = -1$, $b = 2$

PI: $y = 2\sin x - \cos x$

b) CF

$m^2 + 2m + 5 = 0$

$m = -1 \pm 2i$

$y = e^{-x}(A\cos 2x + B\sin 2x) + 2\sin x - \cos x$

$y = 0$ when $x = 0$

$A - 1 = 0$, so $A = 1$

$$\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) + 2\cos x + \sin x$$

$\frac{dy}{dx} = 0$ when $x = 0$

$-A + 2B + 2 = 0$

$B = -\frac{1}{2}$

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$$y = e^{-x}\left(\cos 2x - \frac{1}{2}\sin 2x\right) + 2\sin x - \cos x$$

c) $e^{-x} \to 0$ as $x \to \infty$

$y \approx 2\sin x - \cos x$

amplitude $= \sqrt{(2^2 + 1^2)}$

y oscillates with amplitude $\sqrt{5}$ (period 2π)

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7 The differential equation

$$\frac{d^2y}{dx^2} + 16y = 24\cos 4x$$

is to be solved.

a. Show that a particular integral of the differential equation is $y = 3x \sin 4x$

b. Hence find the general solution of the differential equation.

c. Find the particular solution for which $y = 2$ and $\frac{dy}{dx} = 8$ when $x = 0$

d. Find the exact value of y when $x = \frac{\pi}{8}$ for this particular solution.

(9 marks)

a) Product rule

$$y = 3x \sin 4x$$

$$\frac{dy}{dx} = 3\sin 4x + 12x\cos 4x$$

$$\frac{d^2y}{dx^2} = 12\cos 4x + 12\cos 4x - 48x\sin 4x$$

$$\frac{d^2y}{dx^2} = 24\cos 4x - 48x\sin 4x$$

$$\frac{d^2y}{dx^2} + 16y = 24\cos 4x - 48x\sin 4x + 48x\sin 4x$$

$$\boxed{\frac{d^2y}{dx^2} + 16y = 24\cos 4x \text{ (shown)}}$$

b) CF

$$m^2 + 16 = 0$$

$$m = \pm 4i$$

$$\boxed{y = A\cos 4x + B\sin 4x + 3x\sin 4x}$$

c) $y = 2$ when $x = 0$

$$A = 2$$

$$\frac{dy}{dx} = -4A\sin 4x + 4B\cos 4x + 3\sin 4x + 12x\cos 4x$$

$\frac{dy}{dx} = 8$ when $x = 0$

$$4B = 8$$

$$B = 2$$

$$\boxed{y = 2\cos 4x + 2\sin 4x + 3x\sin 4x}$$

d) $x = \frac{\pi}{8}$: $4x = \frac{\pi}{2}$

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$$y = 2(0) + 2(1) + 3\left(\frac{\pi}{8}\right)(1)$$

$$\boxed{y = 2 + \frac{3\pi}{8}}$$

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8 The differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x}$$

is to be solved.

a. Explain why a particular integral of the form $\lambda x^2 e^{3x}$ is needed, and find the value of λ.

b. Find the particular solution for which $y = 1$ and $\frac{dy}{dx} = 5$ when $x = 0$

c. Show that $y > 0$ for all values of x.

(8 marks)

a) CF

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$\text{CF: } y = (A + Bx)e^{3x}$$

e^{3x} and xe^{3x} are both in the CF, so try $\lambda x^2 e^{3x}$

$$y = \lambda x^2 e^{3x}$$

$$\frac{dy}{dx} = \lambda(2x + 3x^2)e^{3x}$$

$$\frac{d^2y}{dx^2} = \lambda(2 + 6x)e^{3x} + 3\lambda(2x + 3x^2)e^{3x} = \lambda(2 + 12x + 9x^2)e^{3x}$$

Substitute and divide by λe^{3x}

$$(2 + 12x + 9x^2) - 6(2x + 3x^2) + 9x^2 = \frac{6}{\lambda}$$

$$2 = \frac{6}{\lambda}$$

$$\boxed{\lambda = 3}$$

b) $y = (A + Bx)e^{3x} + 3x^2e^{3x}$

$$y = (A + Bx + 3x^2)e^{3x}$$

$y = 1$ when $x = 0$

$$A = 1$$

$$\frac{dy}{dx} = (B + 6x)e^{3x} + 3(A + Bx + 3x^2)e^{3x}$$

$\frac{dy}{dx} = 5$ when $x = 0$

$$B + 3A = 5$$

$$B = 2$$

$$\boxed{y = (1 + 2x + 3x^2)e^{3x}}$$

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c) Discriminant of $3x^2 + 2x + 1$

$b^2 - 4ac = 4 - 12 = -8$

$-8 < 0$ and $3 > 0$, so $3x^2 + 2x + 1 > 0$ for all x

$e^{3x} > 0$ for all x

$y > 0$ for all x (shown)

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9 The differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 13 - 6x + 12e^{-x}$$

is to be solved.

a. Find the general solution of the differential equation.

A particular solution has $y = 0$ when $x = 0$, and for this solution $y - x$ remains finite as $x \to \infty$

b. Find this particular solution.

c. Find the value of $\frac{dy}{dx}$ when $x = 0$ for this particular solution.

(9 marks)

a) CF

$$m^2 + m - 6 = 0$$

$$(m + 3)(m - 2) = 0$$

$$CF: y = Ae^{2x} + Be^{-3x}$$

PI: e^{-x} is not in the CF, so try $y = ax + b + ce^{-x}$

$$\frac{dy}{dx} = a - ce^{-x}$$

$$\frac{d^2y}{dx^2} = ce^{-x}$$

$$ce^{-x} + a - ce^{-x} - 6(ax + b + ce^{-x}) = 13 - 6x + 12e^{-x}$$

Compare coefficients

$$x: -6a = -6, \text{ so } a = 1$$

$$\text{constant: } a - 6b = 13, \text{ so } b = -2$$

$$e^{-x}: -6c = 12, \text{ so } c = -2$$

$$\boxed{y = Ae^{2x} + Be^{-3x} + x - 2 - 2e^{-x}}$$

b) As $x \to \infty$, $e^{-3x} \to 0$ and $e^{-x} \to 0$, but $e^{2x} \to \infty$

$$y - x = Ae^{2x} + Be^{-3x} - 2 - 2e^{-x}$$

$A = 0$ for $y - x$ to stay finite

$y = 0$ when $x = 0$

$$B - 2 - 2 = 0$$

$$B = 4$$

$$\boxed{y = 4e^{-3x} - 2e^{-x} + x - 2}$$

c)

$$\frac{dy}{dx} = -12e^{-3x} + 2e^{-x} + 1$$

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$$\frac{dy}{dx} = -12 + 2 + 1 \text{ at } x = 0$$

$$\boxed{\frac{dy}{dx} = -9}$$

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10 The temperature, θ °C, inside an industrial oven t minutes after it is switched on is modelled by the differential equation

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 10\theta = 1800$$

When the oven is switched on, its temperature is 20 °C and is not changing.

a. Find the general solution of the differential equation.

b. Show that the particular solution is

$$\theta = 180 - e^{-t}\left(160\cos 3t + \frac{160}{3}\sin 3t\right)$$

c. Show that $\frac{d\theta}{dt} = \frac{1600}{3}e^{-t}\sin 3t$

d. Hence find, according to the model, the maximum temperature in the oven, giving your answer to the nearest degree.

e. State the temperature that the oven settles at in the long term, according to the model. (11 marks)

a) CF

$$m^2 + 2m + 10 = 0$$

$$m = \frac{-2 \pm \sqrt{(4 - 40)}}{2} = -1 \pm 3i$$

$$\text{CF: } \theta = e^{-t}(A\cos 3t + B\sin 3t)$$

PI: try $\theta = k$

$$10k = 1800$$

$$k = 180$$

$$\boxed{\theta = 180 + e^{-t}(A\cos 3t + B\sin 3t)}$$

b) $\theta = 20$ when $t = 0$

$$20 = 180 + A$$

$$A = -160$$

$$\frac{d\theta}{dt} = -e^{-t}(A\cos 3t + B\sin 3t) + e^{-t}(-3A\sin 3t + 3B\cos 3t)$$

$\frac{d\theta}{dt} = 0$ when $t = 0$

$$0 = -A + 3B$$

$$0 = 160 + 3B$$

$$B = -\frac{160}{3}$$

$$\boxed{\theta = 180 - e^{-t}\left(160\cos 3t + \frac{160}{3}\sin 3t\right) \text{ (shown)}}$$

c) Product rule

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$$\frac{d\theta}{dt} = e^{-t}(160\cos 3t + \frac{160}{3}\sin 3t) - e^{-t}(-480\sin 3t + 160\cos 3t)$$

$$\frac{d\theta}{dt} = e^{-t}(\frac{160}{3} + 480)\sin 3t$$

$$\boxed{\frac{d\theta}{dt} = \frac{1600}{3}e^{-t}\sin 3t \text{ (shown)}}$$

d) $\frac{d\theta}{dt} = 0$ when $\sin 3t = 0$

$$3t = \pi$$

$$t = \frac{\pi}{3}$$

$\sin 3t$ changes from + to - at $t = \frac{\pi}{3}$, so this is a maximum

$$\theta = 180 - e^{-\pi/3}(160\cos\pi + \frac{160}{3}\sin\pi)$$

$$\theta = 180 + 160e^{-\pi/3}$$

$$\theta = 180 + 56.1...$$

$$\boxed{\text{Maximum temperature } 236\ ^\circ C}$$

e) $e^{-t} \to 0$ as $t \to \infty$

$$\boxed{\theta \to 180\ ^\circ C}$$

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11 A sledge is pulled from rest across horizontal snow. Its displacement, x metres, from its starting point t seconds after it starts to move is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} = 8$$

a. Find the complementary function and explain why a particular integral of the form $x = k$, where k is a constant, will not work.

b. Using a particular integral of the form $x = \lambda t$, find the particular solution for the motion of the sledge.

c. Find, according to the model, the distance travelled by the sledge in the first 3 seconds.

d. Find, according to the model, the speed of the sledge after 0.5 seconds, and state the speed that the sledge approaches.

(10 marks)

a) CF

$$m^2 + 4m = 0$$

$$m(m+4) = 0$$

$$m = 0 \text{ or } m = -4$$

$$\boxed{\text{CF: } x = A + Be^{-4t}}$$

A constant is part of the CF (the term A)

$$\boxed{x = k \text{ gives } 0 + 4(0) = 0, \text{ never } 8}$$

b) Try $x = \lambda t$

$$\frac{dx}{dt} = \lambda, \frac{d^2x}{dt^2} = 0$$

$$0 + 4\lambda = 8$$

$$\lambda = 2$$

$$x = A + Be^{-4t} + 2t$$

$$\frac{dx}{dt} = -4Be^{-4t} + 2$$

Starts from rest at the origin: $x = 0$, $\frac{dx}{dt} = 0$ when $t = 0$

$$A + B = 0$$

$$-4B + 2 = 0$$

$$B = \frac{1}{2}, A = -\frac{1}{2}$$

$$\boxed{x = 2t + \frac{1}{2}e^{-4t} - \frac{1}{2}}$$

c) $t = 3$ (the sledge never turns back, since $\frac{dx}{dt} > 0$)

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$$x = 6 + \frac{1}{2}e^{-12} - \frac{1}{2}$$

$$\boxed{x = 5.50 \text{ m (3 s.f.)}}$$

d)

$$\frac{dx}{dt} = 2 - 2e^{-4t}$$

$$\frac{dx}{dt} = 2 - 2e^{-2} \text{ at } t = 0.5$$

$$\boxed{\text{speed} = 1.73 \text{ m s}^{-1} \text{ (3 s.f.)}}$$

$e^{-4t} \to 0$ as $t \to \infty$

$$\boxed{\text{The speed approaches 2 m s}^{-1}}$$

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12 A particle P moves along the x-axis. At time t seconds, $t \geq 0$, P is x metres from the origin O, where

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 2t + 7$$

a. Find the general solution of the differential equation.

b. Find the particular solution for which $x = 5$ and $\frac{dx}{dt} = -4$ when $t = 0$

c. Show that the particle is momentarily at rest when $e^{-t} = \frac{\sqrt{17} - 1}{8}$

d. Find, according to the model, the minimum distance between P and O, giving your answer to 3 significant figures, and justify that it is a minimum.

e. Describe the motion of P for large values of t.

(12 marks)

a) CF

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$\text{CF: } x = Ae^{-t} + Be^{-2t}$$

PI: try $x = at + b$

$$0 + 3a + 2(at + b) = 2t + 7$$

$$t: 2a = 2, \text{ so } a = 1$$

$$\text{constant: } 3a + 2b = 7, \text{ so } b = 2$$

$$\boxed{x = Ae^{-t} + Be^{-2t} + t + 2}$$

b) $x = 5$ when $t = 0$

$$A + B + 2 = 5$$

$$A + B = 3$$

$$\frac{dx}{dt} = -Ae^{-t} - 2Be^{-2t} + 1$$

$\frac{dx}{dt} = -4$ when $t = 0$

$$-A - 2B + 1 = -4$$

$$A + 2B = 5$$

$$B = 2, A = 1$$

$$\boxed{x = e^{-t} + 2e^{-2t} + t + 2}$$

c) At rest: $\frac{dx}{dt} = 0$

$$-e^{-t} - 4e^{-2t} + 1 = 0$$

Let $u = e^{-t}$

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$$4u^2 + u - 1 = 0$$

$$u = \frac{-1 \pm \sqrt{17}}{8}$$

$u = e^{-t} > 0$, so take the positive root

$$\boxed{e^{-t} = \frac{\sqrt{17} - 1}{8} \text{ (shown)}}$$

d)

$u = 0.39038...$

$t = -\ln u = 0.94061...$

$x = u + 2u^2 + t + 2$

$x = 0.39038 + 0.30480 + 0.94061 + 2$

$$\boxed{\text{minimum distance} = 3.64 \text{ m (3 s.f.)}}$$

$$\frac{d^2x}{dt^2} = e^{-t} + 8e^{-2t}$$

$\frac{d^2x}{dt^2} > 0$ for all t, so this is a minimum

e) $e^{-t} \to 0$ and $e^{-2t} \to 0$

$x \approx t + 2$

$$\boxed{P \text{ moves away from O with speed approaching 1 m s}^{-1}}$$

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