Series with de Moivre's theorem

Answers

Further Maths topic practice

This topic online

Q	Answer	Marks
1	$C + iS = 1 + e^{i\theta} + e^{2i\theta} + \ldots + e^{5i\theta}$, a geometric series with $a = 1$, $r = e^{i\theta}$, 6 terms, so the sum is $(e^{6i\theta} - 1)/(e^{i\theta} - 1)$	3
2	Expanding gives $1 - k(e^{i\theta} + e^{-i\theta}) + k^2 = 1 - 2k\cos\theta + k^2$; multiplying the top and bottom by $1 - (1/2)e^{-i\theta}$ gives $(1 - (1/2)\cos\theta + (1/2)i\sin\theta)/(5/4 - \cos\theta)$, then multiply by 4	4
3	$e^{i\theta} - 1 = e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2}) = 2i\sin(\theta/2)\,e^{i\theta/2}$; $\lvert 2i\rvert = 2$, $\lvert e^{i\theta/2}\rvert = 1$ and $\sin(\theta/2) > 0$ for $0 < \theta < 2\pi$	4
4	**a)** Geometric with $a = 1$, $r = -(1/2)e^{i\theta}$, $\lvert r\rvert = 1/2 < 1$, so $C + iS = 1/(1 + (1/2)e^{i\theta}) = 2/(2 + e^{i\theta})$ **b)** $C = (4 + 2\cos\theta)/(5 + 4\cos\theta)$, $S = -2\sin\theta/(5 + 4\cos\theta)$	6
5	Geometric with $a = e^{i\theta}$, $r = (1/2)e^{2i\theta}$, so $C + iS = e^{i\theta}/(1 - (1/2)e^{2i\theta}) = 2e^{i\theta}/(2 - e^{2i\theta})$; multiplying by $2 - e^{-2i\theta}$ gives $(4e^{i\theta} - 2e^{-i\theta})/(5 - 4\cos 2\theta)$, whose imaginary part is $6\sin\theta/(5 - 4\cos 2\theta)$	7
6	$C + iS = e^{i\theta}(e^{in\theta} - 1)/(e^{i\theta} - 1) = e^{i(n+1)\theta/2}\sin(n\theta/2)/\sin(\theta/2)$, real part gives C; with $n = 3$, $\theta = 2\pi/7$: $C = \cos(4\pi/7)\sin(3\pi/7)/\sin(\pi/7) = -\sin(3\pi/7)\cos(3\pi/7)/\sin(\pi/7) = -(1/2)\sin(6\pi/7)/\sin(\pi/7) = -1/2$	9
7	**a)** $(1 + e^{i\theta})^6 = 1 + 6e^{i\theta} + 15e^{2i\theta} + 20e^{3i\theta} + 15e^{4i\theta} + 6e^{5i\theta} + e^{6i\theta} = C + iS$ **b)** $C = 64\cos^6(\theta/2)\cos 3\theta$, $S = 64\cos^6(\theta/2)\sin 3\theta$ **c)** $\theta = \pi/3, 2\pi/3$	8
8	**a)** $C + iS = \Sigma\, e^{ik\theta}/k! = \Sigma\,(e^{i\theta})^k/k! = e^{e^{i\theta}}$ **b)** $C = e^{\cos\theta}\cos(\sin\theta)$, $S = e^{\cos\theta}\sin(\sin\theta)$ **c)** $\theta = \pi/2$: $C = 1 - 1/2! + 1/4! - \ldots$ and $e^0\cos(1) = \cos 1$	7
9	**a)** Geometric with $\lvert r\rvert = \lvert z\rvert = 2/3 < 1$; sum $1/(1 - z)$ **b)** $1/(1 - z) = (9 - 6\cos\theta + 6i\sin\theta)/(13 - 12\cos\theta)$; imaginary part $6\sin\theta/(13 - 12\cos\theta)$ **c)** Real part $(9 - 6\cos\theta)/(13 - 12\cos\theta)$: numerator ≥ 3 and denominator ≥ 1, so it is never 0 **d)** $\cos\theta = 2/3$	9
10	**a)** Geometric with $a = e^{i\theta}$, $r = e^{2i\theta}$, n terms **b)** $C + iS = e^{in\theta}\sin n\theta/\sin\theta$, so $C = \sin n\theta\cos n\theta/\sin\theta = \sin 2n\theta/(2\sin\theta)$ and $S = \sin^2 n\theta/\sin\theta$ **c)** $\theta = \pi/8, \pi/4, 3\pi/8, \pi/2, 5\pi/8, 3\pi/4, 7\pi/8$	10

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11	**a)** $C + iS = \Sigma\, nCr\, (e^{2i\theta})^r = (1 + e^{2i\theta})^n = (2\cos\theta\, e^{i\theta})^n = 2^n \cos^n\theta\, e^{in\theta}$ **b)** $C = 2^n \cos^n\theta \cos n\theta$, $S = 2^n \cos^n\theta \sin n\theta$ **c)** (i) -27 (ii) 0	9
12	**a)** The kth move (k = 0, 1, 2, ...) has length $(1/2)^k$ and direction $k\theta$, so it is $(1/2)^k e^{ik\theta}$; the position is the sum of the moves **b)** $z = 1/(1 - (1/2)e^{i\theta}) = 2/(2 - e^{i\theta}) = (4 - 2\cos\theta + 2i \sin\theta)/(5 - 4\cos\theta)$ **c)** 0.8 m east and 0.4 m north **d)** $\lvert z\rvert = 2/\lvert 2 - e^{i\theta}\rvert = 2/\sqrt{(5 - 4\cos\theta)}$; greatest 2 m, least 2/3 m **e)** Real part $(4 - 2\cos\theta)/(5 - 4\cos\theta) \geq 2/3 > 0$, so the point is always east of the north line through O	12

Full worked solutions, written out by hand, are on the solutions sheet for this topic.

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